The first compulsory exercise

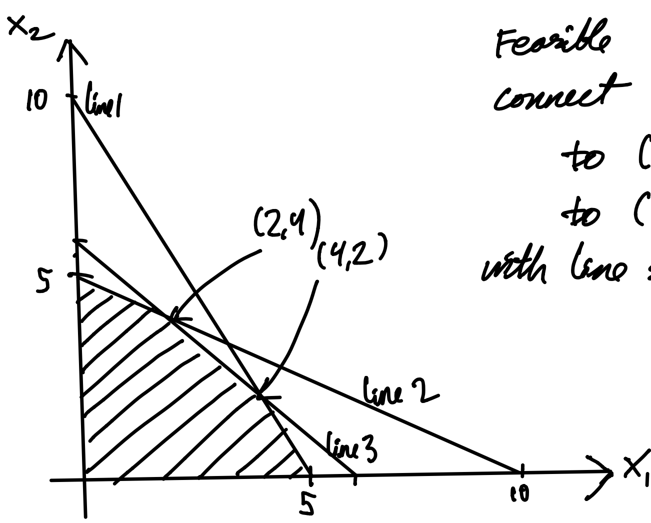
1a) Here: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 10 \\ 10 \\ 6 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

1b) Feasible region:

$2x_1 + x_2 = 10$ (line 1)

$x_1 + 2x_2 = 10$ (line 2)

$x_1 + x_2 = 6$ (line 3)



Feasible region:
connect (0,0) to (5,0) to (4,2) to (2,4) to (0,5) to (0,0) with line segments

c) Let us solve using simplex:

Initial dictionary:

$$\begin{aligned} \eta &= 0 + 2x_1 + 3x_2 \\ w_1 &= 10 - 2x_1 - x_2 \\ w_2 &= 10 - x_1 - 2x_2 \\ w_3 &= 6 - x_1 - x_2 \end{aligned}$$

Large coefficient rule: x_2 is entering

ratios: $\frac{1}{10}$, $\frac{1}{5}$, $\frac{1}{6}$. $\frac{1}{5}$ is biggest, so that w_2 is leaving.

Rewrite the second constraint as $x_2 = 5 - \frac{1}{2}x_1 - \frac{1}{2}w_2$. This gives

$$\begin{aligned}
\eta &= 15 + 0.5x_1 - 1.5w_2 \\
w_1 &= 5 - 1.5x_1 + 0.5w_2 \\
x_2 &= 5 - 0.5x_1 - 0.5w_2 \\
w_3 &= 1 - 0.5x_1 + 0.5w_2
\end{aligned}$$

Now x_1 is entering. Ratios are: 0.3, 0.1, 0.5

0.5 is biggest, so that w_3 is leaving

Rewrite third constraint as $x_1 = 2 - 2w_2 + w_2$. This gives

$$\begin{aligned}
\eta &= 16 - w_3 - w_2 \\
w_1 &= 2 + 3w_3 - w_2 \\
x_2 &= 4 + w_3 - w_2 \\
x_1 &= 2 - 2w_3 + w_2
\end{aligned}$$

This is optimal, with value 16.

The basic solution is: $w_2 = w_3 = 0$, $w_1 = 2$, $x_2 = 4$, $x_1 = 2$

so that $\vec{x} = (2, 4)$ is optimal

d) Change the objective to $2x_1 + 2x_2$

Initial dictionary:

$$\begin{aligned}
\eta &= 0 + 2x_1 + 2x_2 \\
w_1 &= 10 - 2x_1 - x_2 \\
w_2 &= 10 - x_1 - 2x_2 \\
w_3 &= 6 - x_1 - x_2
\end{aligned}$$

We can choose the same entering variable as in c) (x_2)

Same leaving variable, ($x_2 = 5 - \frac{1}{2}x_1 - \frac{1}{2}w_2$)

$$\begin{aligned}
\eta &= 10 + x_1 - w_2 \\
w_1 &= 5 - 1.5x_1 + 0.5w_2 \\
x_2 &= 5 - 0.5x_1 - 0.5w_2 \\
w_3 &= 1 - 0.5x_1 + 0.5w_2
\end{aligned}$$

x_1 is entering also here, so same leaving variable as in c) (w_3). We get ($x_1 = 2 - 2w_3 + w_2$)

$$\eta = 12 - 2w_3$$
$$w_1 = 2 + 3w_3 - w_2$$
$$x_2 = 4 + w_3 - w_2$$
$$x_1 = 2 - 2w_3 + w_2$$

This is optimal, value 12. basic solution $w_2 = w_3 = 0$, $w_1 = 2$, $x_2 = 4$, $x_1 = 2$ $\Rightarrow \vec{x} = (2, 4)$

This is not optimal, since we can increase w_2 and keep 12 as value.

How much can we increase w_2?

- First constraint: Can increase w_2 to 2
- Second constraint: Can increase w_2 to 4
- Third constraint: Can increase w_2 to infinity.

Thus, w_2 can be increased to 2.

Any point $(2+w_2, 4-w_2)$ with $0 \le w_2 \le 2$ is optimal.

Exercise 2

Initial dictionary:

$$\eta = \quad x_1 + x_2 + x_3$$
$$w_1 = 4 - 2x_1 + 2x_2 - x_3$$
$$w_2 = 2 - 3x_1 + x_2 - 2x_3$$

Any x_i can enter

Let us choose x_2. ratios are $-\frac{1}{2}, -\frac{1}{2}$, which are ≤ 0, so that the problem is unbounded.

Exercise 3

a) Write $x_i = x_i^+ - x_i^-$, where $x_i^+, x_i^- \geq 0$

$$\begin{array}{l} \min \ \|x\|_1 \\ \text{s.t. } Ax = p \end{array} \iff \begin{array}{l} \min \ \sum_i (x_i^+ + x_i^-) \\ \text{s.t. } A(x^+ - x^-) = p \\ \qquad x^+, x^- \geq 0 \end{array}$$

(note that $\underbrace{|x| = |x^+ - x^-| \leq |x^+| + |x^-| = x^+}_{} + x^-$

The only way to have equality here is to have one of x^+ and x^- equal to zero

$$A(x^+ - x^-) = [A \quad -A] \begin{bmatrix} x^+ \\ x^- \end{bmatrix}$$

$$Ax = p \iff Ax \leq p \text{ and } -Ax \leq -p$$

we get: $A(x^+ - x^-) = p \iff [A \quad -A] \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \leq p$

$$[-A \quad A] \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \leq -p$$

combining these two:

$$\begin{bmatrix} A & -A \\ -A & A \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \end{bmatrix} \leq \begin{bmatrix} p \\ -p \end{bmatrix}$$

b) Primal dictionary:

$$\begin{array}{rcrrrrrr} \xi & = & & -x_1 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 \\ w_1 & = & 1 & -x_1 & & +x_3 & +x_4 & & -x_6 \\ w_2 & = & 1 & & -x_2 & +x_3 & & +x_5 & -x_6 \\ w_3 & = & -1 & +x_1 & & -x_3 & -x_4 & & +x_6 \\ w_4 & = & -1 & & +x_2 & -x_3 & & -x_5 & +x_6 \end{array}$$

Dictionary is not primal feasible, but it is dual feasible.

Dual dictionary:

$$
\begin{array}{lllllll}
-\eta & = & & -y_1 & -y_2 & +y_3 & +y_4 \\
z_1 & = & 1 & +y_1 & & -y_3 & \\
z_2 & = & 1 & & +y_2 & & -y_4 \\
z_3 & = & 1 & -y_1 & -y_2 & +y_3 & +y_4 \\
z_4 & = & 1 & -y_1 & & +y_3 & \\
z_5 & = & 1 & & -y_2 & & \\
 & & & & & & +y_4 \\
z_6 & = & 1 & +y_1 & +y_2 & -y_3 & -y_4
\end{array}
$$

Choose y_3 as entering, z_1 leaves, and this gives: $y_3 = 1 + y_1 - z_1$

New dictionary:

$$
\begin{array}{llllll}
-\eta & = & 1 & -y_2 & -z_1 & +y_4 \\
y_3 & = & 1 + y_1 & & -z_1 & \\
z_2 & = & 1 & +y_2 & & -y_4 \\
z_3 & = & 2 & -y_2 & -z_1 & +y_4 \\
z_4 & = & 2 & & -z_1 & \\
z_5 & = & 1 & -y_2 & & \\
 & & & & & +y_4 \\
z_6 & = & & y_2 & +z_1 & -y_4
\end{array}
$$

y_4 is entering, z_6 is leaving (degenerate pivot and dictionary)

$y_4 = y_2 + z_1 - z_6$.

New dictionary:

$$
\begin{array}{lllllll}
-\eta & = & 1 & & & & -z_6 \\
y_3 & = & 1 & +y_1 & & -z_1 & \\
z_2 & = & 1 & & & -z_1 & +z_6 \\
z_3 & = & 2 & & & & -z_6 \\
z_4 & = & 2 & & & -z_1 & \\
z_5 & = & 1 & & & +z_1 & -z_6 \\
y_4 & = & & & +y_2 & +z_1 & -z_6
\end{array}
$$

This is optimal!

Primal dictionary:

$$
\begin{array}{lllllllll}
\xi & = & -1 & -w_3 & -x_2 & -2x_3 & -2x_4 & -x_5 & \\
w_1 & = & & -w_3 & & & & & \\
w_2 & = & & & & & & & -w_4 \\
x_1 & = & & w_3 & +x_2 & & +x_4 & -x_5 & -w_4 \\
x_6 & = & 1 & & -x_2 & +x_3 & & +x_5 & +w_4
\end{array}
$$

We see that $w_3 = x_2 = x_3 = x_4 = x_5 = 0$

Also $w_4 = 0$ due to the second constraint.

Therefore: $x_1 = 0$, $x_6 = 1$

$\Rightarrow$ $x^+ = (x_1, x_2, x_3) = (0, 0, 0)$

$x^- = (x_4, x_5, x_6) = (0, 0, 1)$

$\vec{x} = x^+ - x^- = (0,0,0) - (0,0,1) = \underline{(0,0,-1)}$