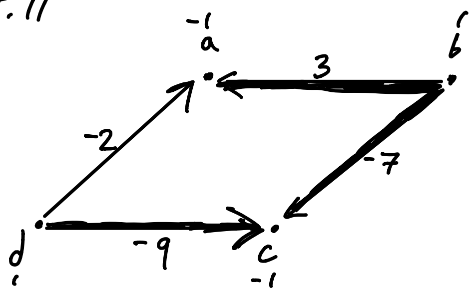
Exercise 14.11



Tree solution x:

flow balance at a: $-x_{ba} = -1 \Rightarrow x_{ba} = 1$

—— " —— b: $x_{ba} + x_{bc} = 1 \Rightarrow x_{bc} = 0$

—— " —— c: $-x_{bc} - x_{dc} = -1 \Rightarrow x_{dc} = 1$

$\vec{x}$ is primal feasible.

Dual variables y: Let a be the root node, so that $y_a = 0$

$y_j - y_i = c_{ij}$

edges in spanning tree:
- (b,a): $y_a - y_b = 3 \Rightarrow y_b = -3$
- (b,c): $y_c - y_b = -7 \Rightarrow y_c = -10$
- (d,c): $y_c - y_d = -9 \Rightarrow y_d = -1$

Dual slack variables z

edge outside spanning tree (d,a): $z_{da} = y_d + c_{da} - y_a = -1 - 2 - 0 = -3$

z is not dual feasible.

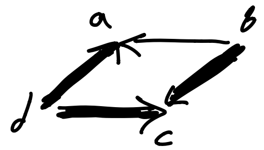
We make a pivot where (d,a) enters the basis,

x_{da} increased to ε:

$$\left.\begin{aligned} x_{da} &= \varepsilon \\ x_{ba} &= 1-\varepsilon \\ x_{bc} &= \varepsilon \\ x_{dc} &= 1-\varepsilon \end{aligned}\right\}$$

can increase ε to 1

(b,a) leaves



Second iteration: $x_{da} = 1$

$x_{ba} = 0$

$x_{bc} = 1$

$x_{dc} = 0$

Dual variables y:

$$\text{in spanning tree} \begin{cases} (d,a): & y_a - y_d = -2 \Rightarrow y_d = 2 \\ (b,c): & y_c - y_b = -7 \Rightarrow y_b = 0 \\ (d,c): & y_c - y_d = -9 \Rightarrow y_c = -7 \end{cases}$$

Dual slack variables:

outside tree: (b,a): $z_{ba} = y_b + c_{ba} - y_a$

$$= 0 + 3 - 0 = 3$$

z dual feasible $\Rightarrow \vec{x}$ is optimal.

Exercise 14.12

Assume that a $\underbrace{\text{square}}_{(|N|-1)\times(|N|-1)}$ submatrix of $\tilde{A}$ is invertible.

I will assume that $1, 2, \dots, |N|-1$ are the columns of this submatrix.

Columns in this submatrix correspond to $|N|-1$ edges.

Assume that those edges do not give a spanning tree.

Then all nodes are not visited by those edges.

1. Assume that the root node (node nr. $|N|$) is not visited by these edges. Then $A_{|N|,\, 1:(|N|-1)}$ is zero, but then the first $|N|-1$ rows also sum to zero (all in total sum to 0).

$\Rightarrow$ $\tilde{A}$ submatrix is not invertible (since all rows sum to 0).

2. Assume that the root node is visited. Then some other node is not visited by the same edges

$\Rightarrow$ some row in the submatrix is zero

$\Rightarrow$ $\tilde{A}$ submatrix is not invertible.

By contradiction, it now follows that the edges provide a spanning tree.

Exercise 14.14

```
remaining_nodes = nodes
added_nodes = ∅
spantree = ∅
connected = true

while remaining_nodes not empty and connected:
    found = false
    for (i,j) in A :
        if i in added_nodes and j in remaining_nodes:
            % add j to the span tree
            % add j to added_nodes, remove it from remaining_nodes.
            found = true
        elif j in added_nodes and i in remaining_nodes:
            % add i to the spantree
            % add i to added_nodes, remove it from remaining_nodes
            found = true
    if not found:
        connected = false
```