Exam 2018

Problem 1

$$\max \quad 5x_1 + 10x_2$$

$$\text{subj. to} \quad \begin{aligned} x_1 + 3x_2 &\le 50 \\ 4x_1 + 2x_2 &\le 60 \\ x_1 &\le 5 \\ x_1, x_2 &\ge 0 \end{aligned}$$

a) Dual problem: $\min \; 50y_1 + 60y_2 + 5y_3$

$$\text{subject to} \quad \begin{aligned} y_1 + 4y_2 + y_3 &\ge 5 \\ 3y_1 + 2y_2 &\ge 10 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

In matrix form: $\max c^T x$ s.t. $Ax \le b$, $x \ge 0$ and $\min b^T y$, $y^T A \ge c$, $y \ge 0$

with $c = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$, $b = \begin{pmatrix} 50 \\ 60 \\ 5 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \\ 1 & 0 \end{pmatrix}$

b) Assume that x is primal feasible, y dual feasible.
Then they are are optimal for their problem if and only if

$$\left.\begin{aligned} x_j z_j &= 0 \qquad j = 1, \dots, n \\ y_i w_i &= 0 \qquad i = 1, \dots, m \end{aligned}\right\} \text{complementary slackness}$$

Assume that $(x_1, x_2) = \underline{(5, 15)}$ is optimal

We have

$$\left.\begin{aligned} x_1 + 3x_2 + w_1 &= 50 \Rightarrow w_1 = 0 \\ 4x_1 + 2x_2 + w_2 &= 60 \Rightarrow \underline{w_2 = 10} \\ x_1 + w_3 &= 5 \Rightarrow w_3 = 0 \end{aligned}\right\} x \text{ is primal feasible}$$

If y is dual feasible and dual optimal $\Rightarrow$ complementary slack.

$$\Rightarrow z_1, z_2 = 0 \qquad (\text{since } x_1 z_1 = x_2 z_2 = 0)$$

$$y_2 = 0 \qquad (y_2 w_2 = 0)$$

$$\Rightarrow \begin{aligned} y_1 + 4y_2 + y_3 &= 5 \\ 3y_1 + 2y_2 &= 10 \end{aligned}$$

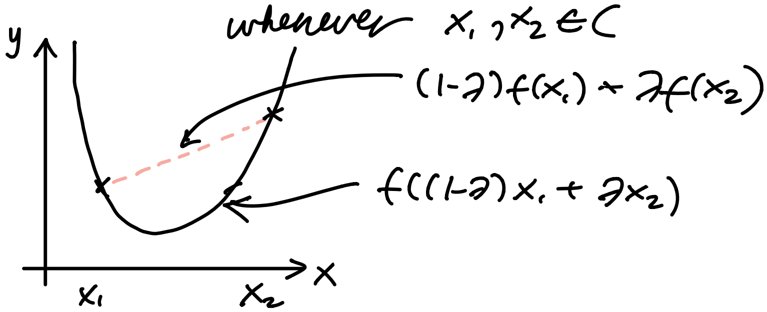
$$\Rightarrow \begin{aligned} y_1 + y_3 &= 5 \quad \Rightarrow y_1 = 5 - \tfrac{10}{3} = \tfrac{5}{3} \\ 3y_1 &= 10 \quad \Rightarrow y_1 = \tfrac{10}{3} \end{aligned}$$

$$\Rightarrow y = \left(\tfrac{10}{3}, 0, \tfrac{5}{3}\right)$$

since $z_1, z_2 = 0$, $y \geq 0$, y is dual feasible.

c) (i) C convex: $(1-\lambda)x_1 + \lambda x_2 \in C$ when $x_1, x_2 \in C$, $\lambda \in [0,1]$

(ii) f convex on C: $f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2)$ whenever $x_1, x_2 \in C$



d) (i) The set of optimal solutions (S) to an LP problem

$$\left.\begin{aligned} \max\ & c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned}\right\} \quad \eta \text{ optimal value}$$

$$S = \{ x \in \mathbb{R}^n \mid c^T x = \eta,\ Ax \leq b,\ x \geq 0 \}$$

Suppose $x_1, x_2 \in S$. Then

1. $c^T((1-\lambda)x_1 + \lambda x_2) = (1-\lambda)c^T x_1 + \lambda c^T x_2 = (1-\lambda)\eta + \lambda\eta = \eta$
2. $A((1-\lambda)x_1 + \lambda x_2) = (1-\lambda)Ax_1 + \lambda Ax_2 \le (1-\lambda)b + \lambda b = b$
3. $(1-\lambda)x_1 + \lambda x_2 \ge 0$ when $x_1, x_2 \ge 0$

It follows that $(1-\lambda)x_1 + \lambda x_2 \in S$, so S is convex.

(ii) If x^1, x^2 are optimal, then so is $(1-\lambda)x_1 + \lambda x_2$, so that there are infinitely many solutions.

Problem 2

Some more on game theory (from previous year's lecture notes)

pure/deterministic strategies in a matrix game

$\underbrace{R}_{\text{row player}}$ chooses a row (i) $\qquad$ $\underbrace{K}_{\text{column player}}$ chooses a column (j)

Note that

$P_R(i) := \max_{j \le n} a_{ij}$: largest payoff for R using strategy i (K should choose this j)

$P_K(j) := \min_{i \le m} a_{ij}$: smallest payoff for K using strategy j (R should choose this i)

$V_* = \max_{j \le n} P_K(j)$: This j gives largest possible payoff to K (knowing that R maximizes his profit)

$V^* = \min_{i \le m} P_R(i)$: This i gives smallest possible payoff from R (knowing that K maximizes his profit)

If $P_K(j) = V_*$: j is called a pure maxmin strategy

$P_R(i) = V^*$: i is called a pure minmax strategy

If $V_* = V^*$ we say that the game has value $V = V^* = V_*$

A pair (r,s) of strategies for R and K is called a saddle point if

$$a_{rj} \leq a_{rs} \leq a_{is} \quad \text{for all } i,j$$

i.e., a_{rs} largest in row r, smallest in column s.

Theorem A game has a value, R has pure minmax strategy r, K has pure maxmin strategy s

$\Updownarrow$

(r,s) is a saddle point

Also, if this holds, then $V = a_{rs}$

Proof: $\Downarrow$

$$\underbrace{a_{is} \geq P_K(s)}_{\text{def. of } P_K(s)} \underbrace{= V_* =}_{s \text{ pure maxmin}} V \underbrace{= V^* = P_R(r)}_{r \text{ pure minmax}} \underbrace{\geq a_{rj}}_{\text{def of } P_R(j)}$$

This holds for all i,j, in particular for r,s, so that

$$a_{rs} \geq V \geq a_{rs} \Rightarrow V = a_{rs}$$

Since $a_{is} \geq a_{rs} \geq a_{rj}$, (r,s) is a saddle point.

$\Uparrow$ If (r,s) is a saddle point:

$$a_{rj} \leq a_{rs} \leq a_{is} \quad \text{for all } i,j$$

We obtain:

$$V_* \overset{\text{def}}{=} \max_j P_K(j) \geq P_K(s) \overset{\text{def}}{=} \underbrace{\min_i a_{is} = a_{rs}}_{\text{since saddle point}}$$

$$V^* \overset{\text{def}}{=} \min_i P_R(i) \leq P_R(r) = \underbrace{\max_j a_{rj} = a_{rs}}_{\text{since saddle point}}$$

It follows that $V_* \geq a_{rs} \geq V^*$, so that $\boxed{V_* \geq V^*}$

Since also
$$P_K(j) \overset{\text{def}}{=} \min_k a_{kj} \leq a_{ij} \leq \max_k a_{ik} = P_R(i),$$
we get that
$$P_K(j) \leq a_{ij} \leq P_R(i) \quad \text{for all } i,j$$
Take max over j:
$$V_* \leq \max_j a_{ij} \leq P_R(i) \quad \text{for all } i$$

Take min over i:
$$V_* \leq \min_i \max_j a_{ij} \leq V^*, \text{ so that } \boxed{V_* \leq V^*}$$

It follows that $V = V^* = V_*$, so that the game has a value, since $a_{rs} = V_* = V^*$, a_{rs} is the value ■

Problem 2

a)
$$A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & -3 & 2 \\ 1 & -2 & -2 \end{pmatrix}$$

pure minmax strategy: $\min_i \max_j a_{ij} = \min_i \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = 1 \quad (r=3)$

pure maxmin strategy: $\max_j \min_i a_{ij} = \max_j (1 \ -3 \ -2) = 1 \quad (s=1)$

The game thus has a value, which is 1

b) payoff matrix for the odd-even game

R chooses	K chooses	sum	
1	1	2	$\Rightarrow$ K wins $\to a_{11} = 2$
1	2	3	$\Rightarrow$ R wins $\to a_{12} = -3$
2	1	3	$\Rightarrow$ R wins $\to a_{21} = -3$
2	2	4	$\Rightarrow$ K wins $\to a_{22} = 4$

This means that $A = \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix}$ (– : payment to row player)

In exam, $A = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$, and we continue with this.

Saddle point of a general game:

A pair (r,s) of strategies for R and K so that

$$a_{rj} \le a_{rs} \le a_{is} \quad \text{for all } i,j$$

(a_{rs} smallest in column, largest in row)

Four possibilities for saddle point:

$(1,1)$: not largest in row 1

$(1,2)$: not smallest in column 2

$(2,1)$: not smallest in column 1

$(2,2)$: not largest in row 2

So, this game does not have a saddle point.

c) Average payoff of row player: $y = \begin{pmatrix} p \\ 1-p \end{pmatrix}$

$$p e_1^T A x + (1-p) e_2^T A x$$

$$= \big((p, 0) + (0, 1-p) \big) A x$$

$$= (p \quad 1-p) \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix} x$$

$$= \big(-2p + 3(1-p) \quad 3p - 4(1-p) \big) x$$

$$= \big(3 - 5p \quad 7p - 4 \big) x$$

average payoff if x chooses 1 : $3 - 5p$

x chooses 2 : $7p - 4$

These are equal when $3 - 5p = 7p - 4 \Leftrightarrow 12p = 7 \Leftrightarrow \underline{p = \frac{7}{12}}$

average payoff: $3-5p = 3 - \frac{35}{12} = \underline{\frac{1}{12}}$

If instead $x = \begin{pmatrix} q \\ 1-q \end{pmatrix}$

average payoff of column player:

$$q y^T A e_1 + (1-q) y^T A e_2 = y^T A \begin{pmatrix} q \\ 1-q \end{pmatrix}$$

$$= y^T \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix} = y^T \begin{pmatrix} -2q + 3(1-q) \\ 3q - 4(1-q) \end{pmatrix}$$

$$= y^T \begin{pmatrix} 3-5q \\ 7q-4 \end{pmatrix}$$

Need now that $3-5q = 7q-4 \iff 12q = 7 \iff q = \frac{7}{12}$

payoff: $3 - 5\frac{7}{12} = \frac{1}{12}$

The game is not fair since expected payoff $\neq 0$.

Problem 3

$$\begin{aligned} \max \quad & x_1 \qquad\quad + 2x_3 \\ \text{subj.to} \quad & x_1 + 2x_2 + x_3 \le 2 \\ & \qquad\qquad\quad x_3 \le 1 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

a)

$$\begin{aligned} \eta &= \qquad\; x_1 \qquad\quad + 2\underset{}{\overset{\text{enters}}{x_3}} \\ w_1 &= 2 - x_1 - 2x_2 - x_3 \\ w_2 &= 1 \qquad\qquad\quad\;\; - x_3 \end{aligned}$$

primal feasible, use primal simplex

ratios: $\frac{1}{2}$

1 $\quad w_2$ leaves

$x_3 = 1 - w_2$

$$\begin{aligned} \eta &= 2 + \overset{\text{enters}}{x_1} \qquad\quad - 2w_2 \\ w_1 &\;\; 1 - x_1 - 2x_2 + w_2 \\ x_3 &= 1 \qquad\qquad\quad - w_2 \end{aligned}$$

ratios: $1 \Rightarrow w_1$ leaves

$x_1 = 1 - w_1 - 2x_2 + w_2$

0

$$\begin{aligned}\eta &= 3 - w_1 - 2x_2 - w_2\\ x_1 &= 1 - w_1 - 2x_2 + w_2\\ x_3 &= 1 \qquad\qquad\quad - w_2\end{aligned}$$

This is optimal!
optimal value: $\underline{3}$
$\vec{x} = (x_1, x_2, x_3) = \underline{(1, 0, 1)}$

b) Assume x^*, y^* feasible for primal and dual problems, with same objective value.

Weak duality $\quad c^T x \leq b^T y \quad$, x primal feasible
y dual feasible.

set $y = y^*$: $\quad c^T x \leq b^T y^* = c^T x^* \Rightarrow x^*$ is optimal.

Similarly y^* is optimal for the dual problem.

c) Dual problem:

$$\begin{aligned}\min \quad & y_1 + 2y_2 + 3y_3\\ \text{subj. to} \quad & y_1 + y_2 \geq 4\\ & y_2 + y_3 \geq 5\\ & y_1 + y_3 \geq 6\\ & y_1, y_2, y_3 \geq 0\end{aligned}$$

$(x_1, x_2, x_3) = (0, 2, 1)$ is clearly primal feasible, and the slacks are $(0, 0, 0)$

$(y_1, y_2, y_3) = \left(\frac{5}{2}, \frac{3}{2}, \frac{7}{2}\right)$ is clearly dual feasible, and the slacks are $(0, 0, 0)$

$\Rightarrow$ so that we must have complementary slack

$$\left(x_i \underbrace{z_i}_{0} = y_i \underbrace{w_i}_{0} = 0 \right)$$

$\Rightarrow$ x is primal optimal, y dual optimal

can also use b), since the objective values are the same:
16 for both the primal and dual.