Exam 2019

Problem 1

$c = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad b = \begin{pmatrix} 40 \\ 70 \\ 10 \end{pmatrix} \quad A = \begin{pmatrix} 2 & 1 \\ 4 & 5 \\ 0 & 1 \end{pmatrix}$

Initial dictionary:

$$\begin{aligned} \eta &= \quad 2x_1 + 3x_2 \quad (\text{enters}) \\ w_1 &= 40 - 2x_1 - x_2 \\ w_2 &= 70 - 4x_1 - 5x_2 \\ \text{leaves} \quad w_3 &= 10 - x_2 \end{aligned}$$

ratios

$\frac{1}{40}$

$\frac{5}{70}$

$\frac{1}{10}$ biggest

w_3 leaves

$x_2 = 10 - w_3$

$$\begin{aligned} \eta &= 30 + 2x_1 - 3w_3 \quad (\text{enters}) \\ w_1 &= 30 - 2x_1 + w_3 \\ \text{leaves} \quad w_2 &= 20 - 4x_1 + 5w_3 \\ x_2 &= 10 - w_3 \end{aligned}$$

ratios

$\frac{2}{30} = \frac{1}{15}$

$\frac{4}{20} = \frac{1}{5}$ biggest, w_2 leaves

0

$x_1 = 5 - \frac{1}{4}w_2 + \frac{5}{4}w_3$

$$\begin{aligned} \eta &= 40 - \tfrac{1}{2}w_2 - \tfrac{1}{2}w_3 \\ w_1 &= 20 + \tfrac{1}{2}w_2 - \tfrac{1}{2}w_3 \\ x_1 &= 5 - \tfrac{1}{4}w_2 + \tfrac{5}{4}w_3 \\ x_2 &= 10 \phantom{- \tfrac{1}{4}w_2} - w_3 \end{aligned}$$

This is optimal

$x_1 = 5$, $x_2 = 10$, $w_1 = 20$

$w_2 = w_3 = 0$

$x = \begin{pmatrix} 5 \\ 10 \end{pmatrix} \quad w = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix} \quad \eta = 40$

b) company A

profit

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{subj. to} \quad & 2x_1 + x_2 \le 40 \\ & 4x_1 + 5x_2 \le 70 \\ & x_2 \le 10 \\ & x_1, x_2 \ge 0 \end{aligned}$$

units of first and second product!

prices for the two products

availability of the three raw materials

a_{ij}: units of raw material i needed to produce one unit of product j

company B: Buys raw material i from company A at price y_i

price: $40y_1 + 70y_2 + 10y_3 = b^T y$

B wants to minimize $b^T y$.

A's price for product j must be more profitable than the price B would sell the needed raw materials for:

prod. 1: $2y_1 + 4y_2 \geq 2$

prod. 2: $y_1 + 5y_2 + y_3 \geq 3$

This amounts to the dual problem.

Dual dictionary, after simplex (negative transpose):

$$
\begin{aligned}
-\xi &= -40 - 20y_1 - 5z_1 - 10z_2 \\
y_2 &= \tfrac{1}{2} - \tfrac{1}{2}y_1 + \tfrac{1}{4}z_1 \\
y_3 &= \tfrac{1}{2} + \tfrac{3}{2}y_1 - \tfrac{5}{4}z_1 + z_2
\end{aligned}
$$

$x_j \Leftrightarrow z_j$

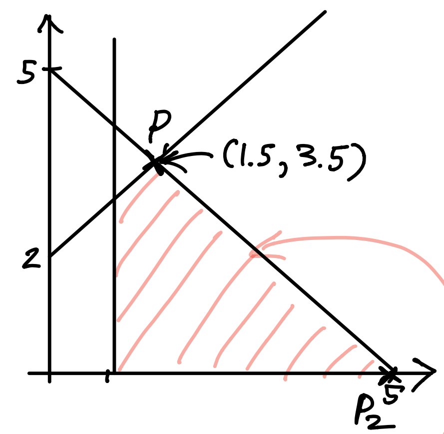
$w_i \Leftrightarrow y_i$

solution: $y_1 = z_1 = z_2 = 0$ $\quad y_2 = y_3 = \frac{1}{2}$

$y = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ $\quad z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad$ objective: $\xi = 40$

Problem 2

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 5 \\ & x_2 - x_1 \le 2 \\ & x_1 \ge 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$



$$\begin{aligned} \xi &= x_1 + x_2 \\ w_1 &= 5 - x_1 - x_2 \\ w_2 &= 2 + x_1 - x_2 \\ w_3 &= -1 + x_1 \end{aligned}$$

not feasible

$x_1 = x_2 = 0 \Rightarrow (0,0)$ is not in feasible region.

all points on this line are optimal.

Basic for the optimal solution:

P_1: nonbasic: w_1, w_2 basic: x_1, x_2, w_3

P_2: nonbasic: x_2, w_1 basic: x_1, w_2, w_3

Cycling doesn't occur since no basic variables are zero: no more than two constraints are met at the same time.

b)

$$\begin{aligned} \xi &= x_1 + x_2 \\ w_1 &= 5 - x_1 - x_2 \\ w_2 &= 2 + x_1 - x_2 \\ w_3 &= -1 + x_1 \end{aligned}$$

$x_1 = 5 - w_1 - x_2$

$$\begin{aligned} \xi &= 5 - w_1 \\ x_1 &= 5 - w_1 - x_2 \\ w_2 &= 7 - w_1 - 2x_2 \\ w_3 &= 4 - w_1 - x_2 \end{aligned}$$

$x_2 = \frac{7}{2} - \frac{1}{2}w_1 - \frac{1}{2}w_2$

$$\begin{aligned}
\xi &= 5 - w_1 \\
x_1 &= \tfrac{3}{2} - \tfrac{1}{2}w_1 + \tfrac{1}{2}w_2 \\
x_2 &= \tfrac{7}{2} - \tfrac{1}{2}w_1 - \tfrac{1}{2}w_2 \\
w_3 &= \tfrac{1}{2} - \tfrac{1}{2}w_1 + \tfrac{1}{2}w_2
\end{aligned}$$

This is both feasible and optimal.

c) Complementary slackness theorem:
if x is primal feasible, y dual feasible, then

$$x \text{ and } y \text{ are jointly optimal}$$
$$\Updownarrow$$
$$x_j z_j = 0 \text{ all } j, \quad y_i w_i = 0 \text{ all } i.$$

Dual problem:
$$\begin{aligned}
\min \quad & 5y_1 + 2y_2 - y_3 \\
\text{s.t.} \quad & y_1 - y_2 - y_3 \geq 1 \\
& y_1 + y_2 \geq 1
\end{aligned}$$

Along the line of optimal points $(x_1 + x_2 = 5)$:

$$x_2 > 0, \quad w_2 > 0, \quad x_1 > 0, \quad w_3 > 0$$
$$\Rightarrow \quad z_2 = 0 \quad y_2 = 0 \quad z_1 = 0 \quad y_3 = 0$$

$$\begin{aligned} y_1 &= 1 \\ y_1 &= 1 \end{aligned} \quad \Rightarrow \quad y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad z = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

objective value: $5 \cdot 1 + 2 \cdot 0 - 0 = \underline{\underline{5}}$

Problem 3

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 2 & 0 & 2 \\ -2 & -1 & 0 \end{pmatrix}$$

a) pure maxmin strategy:

$$\max_j \min_i a_{ij} = \max_j \begin{pmatrix} -2 & -1 & -1 \end{pmatrix} = \underline{\underline{-1}}$$

pure minmax strategy:

$$\min_i \max_j a_{ij} = \min_i \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \underline{0}$$

The two values are different, so the game has no value.

b) Minimax theory for matrix games:

There exist stochastic vectors x^*, y^* so that

$$\max_{\substack{x \geq 0 \\ \mathbf{1}^T x = 1}} (y^*)^T A x = \min_{\substack{y \geq 0 \\ \mathbf{1}^T y = 1}} y^T A x^*$$

(x^*, y^* are mutually optimal)

The cases in a) correspond to x and y being standard basis vectors:

primal in chap. 11: $\max\limits_{x \geq 0} \min\limits_{y \geq 0} y^T A x = \min\limits_{y \geq 0} y^T A x^* \leq (y^*)^T A x^*$

dual in chap. 11: $\min\limits_{y \geq 0} \max\limits_{x \geq 0} y^T A x = \max\limits_{x \geq 0} (y^*)^T A x \geq (y^*)^T A x^*$

Since $\max\limits_j \min\limits_i a_{ij} = \max\limits_j \min\limits_i e_i^T A e_j \overset{\text{see the book}}{\overset{\downarrow}{=}} \max\limits_j \min\limits_{y \geq 0} y^T A e_j \leq \max\limits_{x \geq 0} \min\limits_{y \geq 0} y^T A x$

$\min\limits_i \max\limits_j a_{ij} = \min\limits_i \max\limits_j e_i^T A e_j = \min\limits_i \max\limits_{x \geq 0} e_i^T A x \geq \min\limits_{y \geq 0} \max\limits_{x \geq 0} y^T A x$

since those are different, e_i and e_j can't be mutually optimal.

c) $x^* = (0, \frac{1}{4}, \frac{3}{4})$ $\qquad y^* = (\frac{1}{4}, 0, \frac{3}{4})$

$$\min_{y\geq 0} y^TAx^* = \min_{y\geq 0} y^T \begin{pmatrix} 2 & 2 & -1 \\ 2 & 0 & 2 \\ -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} = \min_{y\geq 0} y^T \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{2} \\ -\frac{1}{4} \end{pmatrix} = \underline{-\frac{1}{4}}$$

$$\max_{x\geq 0} (y^*)^TAx = \max_{x\geq 0} \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 2 & 2 & -1 \\ 2 & 0 & 2 \\ -2 & -1 & 0 \end{pmatrix} x = \max_{x\geq 0} \begin{pmatrix} -1 & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix} x = \underline{-\frac{1}{4}}$$

$-\frac{1}{4}$

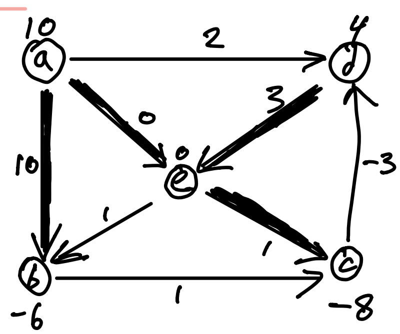
since

$$\left.\begin{array}{ccc} \max\limits_{x\geq 0} \min\limits_{y\geq 0} y^TAx & \geq & \min\limits_{y} y^TAx^* \\ \| & & \| \\ \min\limits_{y\geq 0} \max\limits_{x\geq 0} y^TAx & \leq & \max\limits_{x\geq 0} (y^*)^TAx \end{array}\right\}$$

all four must be equal. so x^*, y^* are mutually optimal.

value of the game is $-\frac{1}{4}$. Since this is negative there is an expected payoff to the row player, who ~~therefore~~ wins in the long run.

Problem 4



spanning tree: visits all nodes, and no cycles, this is easily seen to be the case here.

primal flow:

Flow balance at

ⓒ: $-x_{ec} = -8 \Rightarrow \underline{x_{ec} = 8}$

ⓓ: $\underline{x_{de} = 4}$

ⓑ: $-x_{ab} = -6 \Rightarrow \underline{x_{ab} = 6}$

ⓐ: $x_{ae} + x_{ab} = 10 \Rightarrow \underline{x_{ae} = 10 - 6 = 4}$

Dual variables: $y_j - y_i = c_{ij}$ for (i,j) in spanning tree

Choose a as the root, so that $y_a = 0$.

(a,b) : $y_b - y_a = 10 \Rightarrow \underline{y_b = 10}$

(a,e) : $y_e - y_a = 0 \Rightarrow \underline{y_e = 0}$

(d,e) : $y_e - y_d = 3 \Rightarrow \underline{y_d = -3}$

(e,c) : $y_c - y_e = 1 \Rightarrow \underline{y_c = 1}$

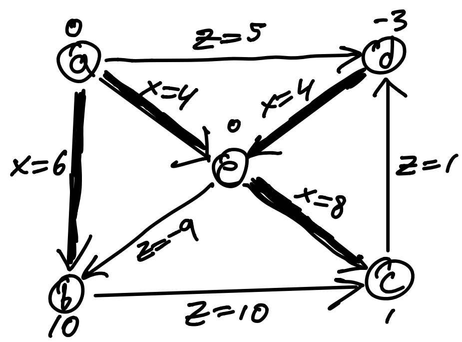
Dual slack variables $z_{ij} = y_i + c_{ij} - y_j$ for (i,j) not in tree

(a,d) : $z_{ad} = y_a + c_{ad} - y_d = 0 + 2 + 3 = \underline{5}$

(e,b) : $z_{eb} = y_b + c_{eb} - y_b = 0 + 1 - 10 = \underline{-9}$

(b,c) : $z_{bc} = y_b + c_{bc} - y_c = 10 + 1 - 1 = \underline{10}$

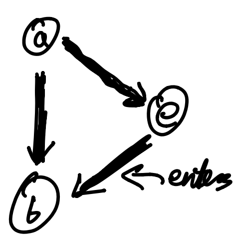
(c,d) : $z_{cd} = y_c + c_{cd} - y_d = 1 - 3 + 3 = \underline{1}$



b) Solution from a) is dual infeasible since $z_{eb} < 0$

Its dual variable is x_{eb}. Take this into the basis.

We then get a cycle :



increase flow on (e,b) to ε

$x_{eb} = \varepsilon$

$x_{ae} = 4 + \varepsilon$

$x_{ab} = 6 - \varepsilon$

ε can be increased to 6,

x_{ab} leaves

$x_{eb} = 6$

$x_{ae} = 10$

$x_{ab} = 0$

The other flows do not change.

$y_j - y_i = c_{ij}$

Dual variables:

(a,e): y_e will not change $\underline{0}$

(d,e): y_d will not change $\underline{-3}$

(e,c): y_c will not change $\underline{1}$

(e,b): $y_b - y_e = 1 \Rightarrow \underline{y_b = 1}$

$z_{ij} = y_i + c_{ij} - y_j$

Dual slack variables

y_b goes from 10 to 1:

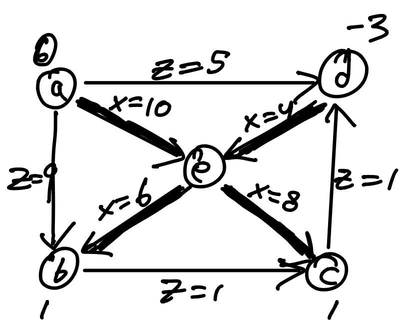
(a,d): does not change $z_{ad} = \underline{5}$

(a,b): $y_a + c_{ab} - y_b = 0 + 9 = \underline{9}$

(b,c): $y_b + c_{bc} - y_c = 10 - 9 = \underline{1}$

(c,d): does not change $z_{cd} = \underline{1}$

This is dual feasible, so optimal.



optimal value: $c_{ae}x_{ae} + c_{eb}x_{eb} + c_{de}x_{de} + c_{ec}x_{ec}$

$= 0 \cdot 10 + 1 \cdot 6 + 3 \cdot 4 + 1 \cdot 8$

$= 6 + 12 + 8 = \underline{\underline{26}}$