Exam 2020

Problem 1

$$\max \quad -x_1 + 3x_2$$
$$\text{s.t} \quad -x_1 + x_2 \le 1$$
$$x_1 \le 4$$
$$x_2 \le 3$$
$$x_1, x_2 \ge 0$$

a) initial feasible solution $(0,0)$: x_1, x_2 are nonbasic.

entering

leaves

$$\begin{aligned} \zeta &= -x_1 + 3x_2 \\ w_1 &= 1 + x_1 - x_2 \\ w_2 &= 4 - x_1 \\ w_3 &= 3 - x_2 \end{aligned}$$

ratios

$1 \rightarrow$ biggest, w_1 leaves

0

$x_2 = 1 + x_1 - w_1$

$\frac{1}{3}$

enters

$$\begin{aligned} \zeta &= 3 + 2x_1 - 3w_1 \\ x_2 &= 1 + x_1 - w_1 \\ w_2 &= 4 - x_1 \\ w_3 &= 2 - x_1 + w_1 \end{aligned}$$

ratios:

-1

$\frac{1}{4}$

$\frac{1}{2} \rightarrow$ biggest, w_3 leaves

$x_1 = 2 - w_3 + w_1$

$$\begin{aligned} \zeta &= 7 - 2w_3 - w_1 \\ x_2 &= 3 - w_3 \\ w_2 &= 2 + w_3 - w_1 \\ x_1 &= 2 - w_3 + w_1 \end{aligned}$$

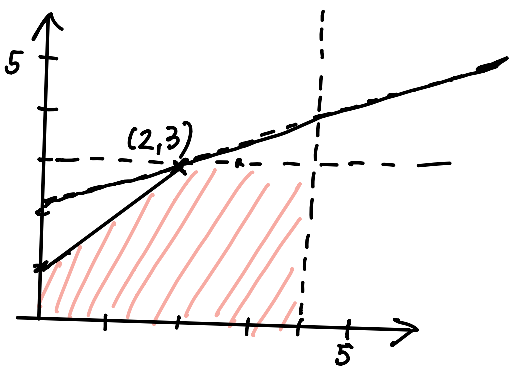
Dictionary is optimal!

$x_1 = 2$, $x_2 = 3$

$w_1 = w_3 = 0$, $w_2 = 2$

$x = (2,3)$ is optimal, with optimal value 7

b)



$-x_1 + x_2 \le 1$

$x_2 \le 1 + x_1$

contour of objective: $-x_1 + 3x_2 = c$

$$x_2 = \frac{1}{3}x_1 + \frac{1}{3}c$$

Problem 2

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 + 4x_3 \\ \text{s.t.} \quad & 2x_1 - 5x_2 = 6 \\ & -x_1 + 3x_3 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\downarrow$

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 + 4x_3 \\ \text{s.t.} \quad & 2x_1 - 5x_2 \le 6 \\ & -2x_1 + 5x_2 \le -6 \\ & x_1 - 3x_3 \le -4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

write $x_3 = y_3 - y_4$, $y_3, y_4 \geq 0$:

$$\max\ 3x_1 + 2x_2 + 4y_3 - 4y_4$$

$$\begin{aligned} \text{s.t.}\quad & 2x_1 - 5x_2 && \le 6 \\ & -2x_1 + 5x_2 && \le -6 \\ & x_1 \qquad -3y_3 + 3y_4 && \le -4 \\ & x_1, x_2, y_3, y_4 \ge 0 \end{aligned}$$

Here the initial basic solution is infeasible, so that we need to apply the two-phase simplex method.

Problem 3

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & Ax \le b \\ & x \ge 0 \end{aligned} \qquad \begin{aligned} w &= b - Ax \\ b &= w + Ax \end{aligned}$$

$$\begin{aligned} \min\ & b^T y \\ \text{s.t.}\ & A^T y \ge c \\ & y \ge 0 \end{aligned} \qquad \begin{aligned} z &= A^T y - c \\ c &= A^T y - z \end{aligned}$$

a) to prove (1):

$$\sum_{i=1}^{m} b_i y_i - \sum_{j=1}^{n} c_j x_j = b^T y - c^T x$$

$$= (w + Ax)^T y - (A^T y - z)^T x$$

$$= (w^T + x^T A^T) y - (y^T A - z^T) x$$

$$x^T A^T y \overset{\shortparallel}{} (x^T A^T y)^T = y^T A x$$

$$= w^T y + \underbrace{x^T A^T y - y^T A x}_{0} + z^T x$$

$$= w^T y + z^T x$$

$$= \sum_{i=1}^{m} w_i y_i + \sum_{j=1}^{n} z_j x_j$$

all terms ≥ 0 when x, y feasible,
so this is ≥ 0

$$\Rightarrow b^T y - c^T x \geq 0$$

$$\Rightarrow \underbrace{b^T y \geq c^T x}_{\text{weak duality}}$$

b) Complementary slackness theorem:

If x is feasible for (P), y feasible for (D), then

$$x_j z_j = 0 \quad j = 1, \dots, n, \qquad y_i w_i = 0 \quad i = 1, \dots, m$$

$$\Updownarrow$$

x, y are optimal for (P) and (D).

We will prove this using (1).

$\Downarrow$ We assume that $w^T y + x^T z = 0$,
(1) then says that

$$\underbrace{\sum_{i=1}^{m} b_i y_i^* \leq \sum_{i=1}^{m} b_i y_i = \sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} c_j x_j^*}_{\geq \text{, due to weak duality.}}$$

This means that we have equality above,
so x, y are also optimal.

$\Uparrow$ insert x^* and y^* in (1): Use strong duality.

$$\overset{\text{strong duality}}{0 \overset{\downarrow}{=}} b^T y^* - c^T x^* \overset{(1)}{=} \underbrace{w^T y^* + z^T x^*}_{\text{all terms} \geq 0} \Rightarrow \underbrace{\begin{array}{ll} w_i y_i^* = 0 & \text{all } i \\ z_j x_j^* = 0 & \text{all } j. \end{array}}_{\text{so, we have complementary slack.}}$$

c) Problem 1: $x = (2,3) \Rightarrow z_1 = z_2 = 0$ (compl. slack)

$w = (0,2,0) \Rightarrow y_2 = 0$ (compl. slack)

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^T = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$A^T y \geq c \Rightarrow A^T y = c$ (if y is optimal)

$$\begin{aligned} -y_1 + y_2 \quad &= -1 \\ y_1 \quad + y_3 &= 3 \end{aligned}$$

$$\Downarrow \quad y_2 = 0$$

$$\begin{aligned} -y_1 \quad &= -1 \\ y_1 + y_3 &= 3 \end{aligned} \Rightarrow \begin{aligned} y_1 &= 1 \\ y_3 &= 2 \end{aligned}$$

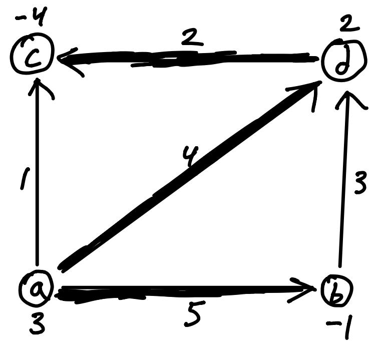
$\Rightarrow \underline{y = (1,0,2)}$ $\quad z = 0 \quad$ optimal value 7.

Alternatively: Dual optimal dictionary is: ($x_j \leftrightarrow z_j$, $w_i \leftrightarrow y_i$)

$$\begin{aligned} -\eta &= -7 - 3z_2 - 2y_2 - 2z_1 \\ y_3 &= 2 + z_2 - y_2 + z_1 \\ y_1 &= 1 \quad + y_2 - z_1 \end{aligned} \Rightarrow \begin{array}{l} y_2 = 0 \\ z_2 = z_1 = 0 \\ y_3 = 2,\ y_1 = 1 \end{array}$$

$$\Rightarrow y = (1,0,2)$$

Problem 4



a) Set $T_1 = \{(a,b), (a,d), (d,c)\}$

Tree solution: Flow balance at ©: $-X_{dc} = -4 \Rightarrow \underline{X_{dc} = 4}$

ⓑ $-X_{ab} = -1 \Rightarrow \underline{X_{ab} = 1}$

ⓐ $X_{ab} + X_{ad} = 3 \Rightarrow \underline{X_{ad} = 3-1 = 2}$

This gives the corresponding tree solution, which is feasible.

b) Dual variables: Use ⓐ as root, so that $y_a = 0$

edges in T_1:

(a,b) : $y_b - y_a = C_{ab} \Rightarrow \underline{y_b = 5}$

(a,d) : $y_d - y_a = C_{ad} \Rightarrow \underline{y_d = 4}$

(d,c) : $y_c - y_d = C_{dc} \Rightarrow y_c = 4+2 = \underline{6}$

Dual slack variables: $z_{ij} = y_i + c_{ij} - y_j$

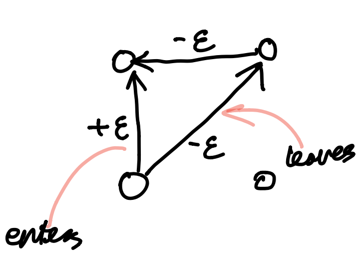
edges not in T_1:

(a,c) : $z_{ac} = y_a + C_{ac} - y_c = 0 + 1 - 6 = \underline{-5}$

(b,d) : $z_{bd} = y_b + C_{bd} - y_d = 5 + 3 - 4 = \underline{4}$

since $z_{ac} < 0$, x is not optimal.

Let X_{ac} enter the basis, increase X_{ac} to ε:



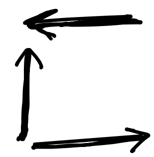
$\tilde{X}_{ac} = \varepsilon$

$\tilde{X}_{ad} = X_{ad} - \varepsilon = 2 - \varepsilon$

$\tilde{X}_{dc} = X_{dc} - \varepsilon = 4 - \varepsilon$

See that we can increase ε to 2. X_{ad} becomes zero, so that (a,d) leaves.

$T_2 = \{(a,c), (d,c), (a,b)\}$



new tree solution: $X_{ac} = 2$

$X_{dc} = 2$

$X_{ab} = 1$ (as before)

Dual variables $\quad y_a = 0$

edges in T_2:

$(a,b): \; y_b - y_a = C_{ab} \Rightarrow \underline{y_b = 5}$ (as before)

$(a,c): \; y_c - y_a = C_{ac} \Rightarrow y_c = y_a + C_{ac} = \underline{1}$

$(d,c): \; y_c - y_d = C_{dc} \rightarrow y_d = y_c + C_{dc} = 1 - 1 = \underline{0}$

Dual slack variables: $\quad z_{ij} = y_i + C_{ij} - y_j$

edges not in T_2:

$(a,d): \; z_{ad} = y_a + C_{ad} - y_d = 0 + 4 - 0 = \underline{4}$

$(b,d): \; z_{bd} = y_b + C_{bd} - y_d = 5 + 3 - 0 = \underline{8}$

Since $z_{ij} > 0$, this is dual feasible, so x is optimal.

(i.e, $X_{ac} = 2$, $X_{dc} = 2$, $X_{ab} = 1$ is optimal flow)

objective value:

$$C_{ac} X_{ac} + C_{dc} X_{dc} + C_{ab} X_{ab}$$

$$= 1 \cdot 2 + 2 \cdot 2 + 5 \cdot 1 = 2 + 4 + 5 = \underline{11}$$