Exam 2017

Problem 1

$$\max \quad x_1 + 2x_2$$
$$\text{s.t.} \quad -x_1 + 3x_2 \le 3$$
$$2x_1 + x_2 \le 8$$
$$x_1, x_2 \ge 0$$

a)

enters

leaves

$$\eta = \quad x_1 + 2x_2$$
$$w_1 = 3 + x_1 - 3x_2$$
$$w_2 = 8 - 2x_1 - x_2$$

ratios

$1 \Rightarrow w_1$ leaves

$\frac{1}{8}$ $\quad x_2 = 1 + \frac{1}{3}x_1 - \frac{1}{3}w_1$

enters

$$\eta = 2 + \frac{5}{3}x_1 - \frac{2}{3}w_1$$
$$x_2 = 1 + \frac{1}{3}x_1 - \frac{1}{3}w_1$$
$$w_2 = 7 - \frac{7}{3}x_1 + \frac{1}{3}w_1$$

ratios

$-\frac{1}{3}$

$\frac{1}{3} \Rightarrow w_2$ leaves

$x_1 = 3 - \frac{3}{7}w_2 + \frac{1}{7}w_1$

$$\eta = 7 - \frac{5}{7}w_2 - \frac{3}{7}w_1$$
$$x_2 = 2 - \frac{1}{7}w_2 - \frac{2}{7}w_1$$
$$x_1 = 3 - \frac{3}{7}w_2 + \frac{1}{7}w_1$$

optimal!

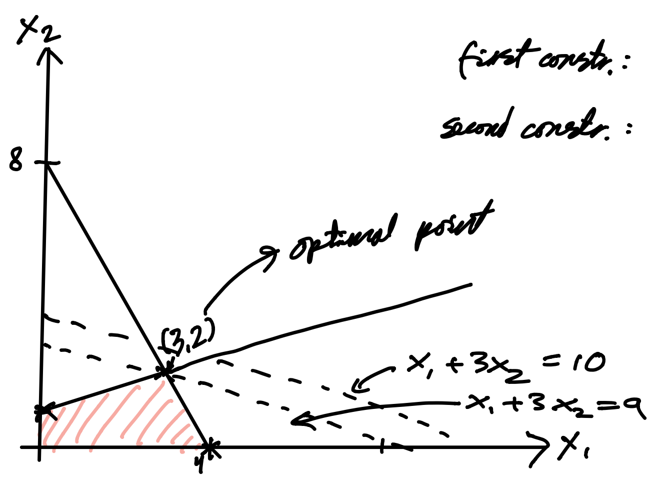
$x_1 = 3$, $x_2 = 2$, $w_1 = w_2 = 0$

$\Rightarrow \underline{x = (3,2)}$ is optimal, with $\underline{\eta = 7}$ optimal value

b)

first constr.: $x_2 \le \frac{1}{3}x_1 + 1$

second constr.: $x_2 \le 8 - 2x_1$



c/ (i) add the constraint $x_1 + 3x_2 \leq 10$ $\rightarrow x_2 \leq -\frac{1}{3}x_1 + \frac{10}{3}$

we check for $(3,2)$: $3 + 3\cdot 2 = 9 \leq 10$

$(0,1)$: $0 + 3\cdot 1 = 3 \leq 10$

$(4,0)$: $4 + 3\cdot 0 = 4 \leq 10$

This means that $x_1 + 3x_2 \leq 10$ is redundant (it is automatically satisfied for the feasible region of (P)

$\Rightarrow$ same optimal point $(3,2)$, optimal value 7.

(ii) since $x_1 + 3x_2 = 9$ for $x = (3,2)$, it is clear that this is optimal for the new objective as well, with 9 as optimal value.

Alternatively: 1/ run simplex again

2/ check the vertices $(0,1)$, $(4,0)$, $(0,0)$ as well.

d/ Dual problem:

$$\begin{aligned} \min \quad & 3y_1 + 8y_2 \\ \text{s.t.} \quad & -y_1 + 2y_2 \geq 1 \\ & 3y_1 + y_2 \geq 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

$(3,2)$ is optimal for (P) with slacks 0 and 0

complementary slack: $z_1 = z_2 = 0$

This implies that $\begin{aligned} -y_1 + 2y_2 &= 1 \\ 3y_1 + y_2 &= 2 \end{aligned} \Rightarrow -7y_1 = -3$

$y_1 = \frac{3}{7}$

$2y_2 = 1 + \frac{3}{7} = \frac{10}{7} \Rightarrow y_2 = \frac{5}{7}$

$\Rightarrow y = (\frac{3}{7}, \frac{5}{7})$ objective: $3\cdot\frac{3}{7} + 8\cdot\frac{5}{7} = \frac{9+40}{7} = 7$

Alternatively, optimal dual dictionary is:

$$\begin{aligned} -\xi &= -7 && -2z_2 && -3z_1 \\ y_2 &= \tfrac{5}{7} && +\tfrac{1}{7}z_2 && +\tfrac{3}{7}z_1 \\ y_1 &= \tfrac{3}{7} && +\tfrac{2}{7}z_2 && -\tfrac{1}{7}z_1 \end{aligned}$$

$z_1 = z_2 = 0 \Rightarrow y_1 = \frac{3}{7}, \; y_2 = \frac{5}{7}$

$\Rightarrow y = \left(\frac{3}{7}, \frac{5}{7}\right)$

Problem 2

a) payoff from R to K is $y^T A x$

(i) if K uses x: R's best strategy is $\min\limits_{\substack{y \geq 0 \\ 1^T y = 1}} y^T A x$

(ii): R uses the strategy from (i): K should find x^* that maximizes payoffs to him:

R chooses y as $\min y^T A x$

K chooses x that maximizes this, i.e..

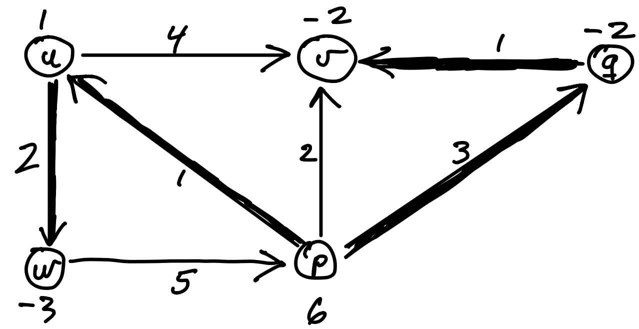
$$\max_{\substack{x \geq 0 \\ 1^T x = 1}} \min_{\substack{y \geq 0 \\ 1^T y = 1}} y^T A x$$

b) $y^T A x$ achieves its minimum y at one of the vertices of $P = \left\{ \begin{matrix} 1^T y = 1 \\ y \geq 0 \end{matrix} \right.$, these are $\vec{e}_i$

$$\Rightarrow \min_{\substack{y \geq 0 \\ 1^T y = 1}} y^T A x = \min_i e_i^T A x$$

$$\max_{\substack{x \geq 0 \\ 1^T x = 1}} \min_{\substack{y \geq 0 \\ 1^T y = 1}} y^T A x = \max_{\substack{x \geq 0 \\ 1^T x = 1}} \min_i e_i^T A x = \max \; \sigma \quad \begin{matrix} \sigma \leq e_i^T A x, \text{ all } i \\ 1^T x = 1 \\ x \geq 0 \end{matrix}$$

Problem 3



a) Flow balance at

(u): $X_{uw} + X_{uv} - X_{pu} = 1$

(v): $-X_{uv} - X_{pv} - X_{qv} = -2$

(w): $X_{wp} - X_{uw} = -3$

(p): $X_{pq} + X_{pv} + X_{pu} - X_{wp} = 6$

(q): $X_{qv} - X_{pq} = -2$

$T_1 = \{ (u,w), (p,u), (p,q), (q,v) \}$

Tree solution: (w) $-X_{uw} = -3 \Rightarrow \underline{X_{uw} = 3}$

(u) $X_{uw} - X_{pu} = 1 \Rightarrow X_{pu} = 3-1 = \underline{2}$

(p) $X_{pu} + X_{pq} = 6 \Rightarrow X_{pq} = 6-2 = \underline{4}$

(v) $-X_{qv} = -2 \Rightarrow \underline{X_{qv} = 2}$

This is feasible.

b) We apply network simplex method:

Dual variables: $y_j - y_i = c_{ij}$ for $(i,j) \in T_1$

choose u as root, so that $y_u = 0$.

$(i,j)\in T_1$:

(u,w): $y_w - y_u = c_{uw} \Rightarrow \underline{y_w = 2}$

(p,u): $y_u - y_p = c_{pu} \Rightarrow \underline{y_p = -1}$

(p,q): $y_q - y_p = c_{pq} \Rightarrow y_q = y_p + c_{pq} = -1 + 3 = \underline{2}$

(q,v): $y_v - y_q = c_{qv} \Rightarrow y_v = y_q + c_{qv} = 2 + 1 = \underline{3}$

Dual slack variables: $z_{ij} = y_i + c_{ij} - y_j$ for $(i,j) \notin T_1$:

$(i,j)\notin T_1$:

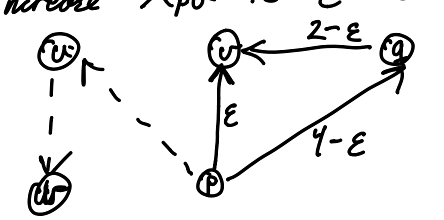
(w,p): $z_{wp} = y_w + c_{wp} - y_p = 2 + 5 + 1 = \underline{8}$

(u,v): $z_{uv} = y_u + c_{uv} - y_v = 0 + 4 - 3 = \underline{1}$

(p,v): $z_{pv} = y_p + c_{pv} - y_v = -1 + 2 - 3 = \underline{-2}$

We have dual infeasibility. Let x_{pv} enter since $z_{pv} < 0$

Increase x_{pv} to $\varepsilon > 0$:



$\tilde{x}_{pv} = \varepsilon$

$\tilde{x}_{pq} = 4 - \varepsilon$

$\tilde{x}_{qv} = 2 - \varepsilon$

x_{qv} becomes zero first, for $\varepsilon = 2$. x_{qv} leaves

new values:

$x_{pv} = 2$

$x_{pq} = 2$

$x_{qv} = 0$

other x_{ij} don't change, i.e.,

$x_{pu} = 2$

$x_{uw} = 3$

new spanning tree:

$$T_2 = \{(u,w), (p,u), (p,q), (p,v)\}$$

Dual variables: y_w, y_u, y_p do not change:
when we use $y_j - y_i = c_{ij}$ (for $(i,j)\in T_2$), we have an edge (i,j) so that one of them is outside the cycle.

Only left to compute y_v: (p,v): $y_v - y_p = c_{pv} \Rightarrow y_v = y_p + 2 = -1 + 2 = \underline{1}$

Only change: y_v has been decreased with 2.

Dual slack variables:

$$(i,j) \notin T_2 \begin{cases} (w,p): & z_{wp} = y_w + c_{wp} - y_p = 8 \quad \text{(does not change)} \\ (u,v): & z_{uv} = y_u + c_{uv} - y_v = z_{uv} + \underline{2} = 1+2 = \underline{3} \\ (q,v): & z_{qv} = y_q + c_{qv} - y_v = 0 + \underline{2} = \underline{2} \end{cases}$$

This is dual feasible, so that it is optimal.

optimal objective value: $c_{uw} x_{uw} + c_{pu} x_{pu} + c_{pq} x_{pq} + c_{pv} x_{pv}$

$$= 2 \cdot 3 + 1 \cdot 2 + 3 \cdot 2 + 2 \cdot 2$$

$$= 6 + 2 + 6 + 4 = \underline{18}$$

c) (i): If optimal solution in a network flow problem is unique, and the b_i are integers:
start with a tree solution, and apply flow balance equations, starting at a leaf node: This gives a value x_{ij} which also must be an integer.
In particular, the optimal tree solution gives a flow which consists of integers.

(ii) If there are two optimal solutions x^1 and x^2, then so are $(1-\lambda) x^1 + \lambda x^2$.
We can choose λ so that this is not integerbased, even if x^1, x^2 are integers.