Second compulsory exercise

Exercise 1

(1) :

$$\min \quad 3x_1 + 5x_2 - x_3$$
$$x_1 - x_2 + x_3 \leq 3$$
$$2x_1 - 3x_2 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

$$\max \quad -3x_1 - 5x_1 + x_3$$
$$x_1 - x_2 + x_3 \leq 3$$
$$2x_1 - 3x_2 \leq 4$$
$$x_1, x_2, x_3 \geq 0$$

$$\Downarrow$$

$$\min \quad 3y_1 + 4y_2$$
$$y_1 + 2y_2 \geq -3$$
$$-y_1 - 3y_2 \geq -5$$
$$y_1 \geq 1$$
$$y_1, y_2 \geq 0$$

$$\max \quad -3y_1 - 4y_2$$
$$y_1 + 2y_2 \geq -3$$
$$-y_1 - 3y_2 \geq -5$$
$$y_1 \geq 1$$
$$y_1, y_2 \geq 0$$

(2)

$$\begin{aligned} \max\quad & 3x_1 + 2x_2 \\ \text{s.t.}\quad & 4x_1 + 2x_2 \leq 16 \\ & x_1 + 2x_2 \leq 8 \\ & x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \min\quad & 16y_1 + 8y_2 + 5y_3 \\ & 4y_1 + y_2 + y_3 \geq 3 \\ & 2y_1 + 2y_2 + y_3 \geq 2 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

b) General form for primal problem:

$$\begin{aligned} \max\quad & c^T x \\ \text{s.t.}\quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

For (1) we have (max $\Leftrightarrow$ min):

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad c = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

For (2) we have

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 16 \\ 8 \\ 5 \end{pmatrix} \quad c = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Dual problem for (2):

$$\begin{aligned} \min\quad & b^T y \\ \text{s.t.}\quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

c) insert x^* in the left hand sides of the inequalities in (2): we get 16, 7, and 5,
with slacks 0, 1, and 0

insert y^* in the left hand sides of the inequalities in the dual of (2): we get 3, and 2
with slacks 0 and 0

since all slacks are nonnegative, x^* is feasible for (2), y^* is feasible for the dual of (2).

We check complementary slack: $x_i z_i = 0$ since $z_1 = z_2 = 0$

$y_i w_i$: $y_1 w_1 = 0$
$y_2 w_2 = 0 \cdot 1 = 0$
$y_3 w_3 = 0$

$\Rightarrow x^*$ is optimal for (P), y^* for (D).

You can also check that x^* and y^* have the same objective value (13) for (P) and (D)

Exercise 2

$f(x), g(x)$ are convex functions.
Prove that $h(x) = \max\{f(x), g(x)\}$ is also convex.

$f(y) \le h(y)$
$f(x) \le h(x)$

$$f((1-\lambda)x+\lambda y) \le (1-\lambda)f(x) + \lambda f(y) \le (1-\lambda)h(x) + \lambda h(y)$$

$g(x) \le h(x)$
$g(y) \le h(y)$

$$g((1-\lambda)x+\lambda y) \le (1-\lambda)g(x) + \lambda g(y) \le (1-\lambda)h(x) + \lambda h(y)$$

$$\Downarrow$$

$$\max\left(f((1-\lambda)x+\lambda y), g((1-\lambda)x+\lambda y)\right) \le (1-\lambda)h(x) + \lambda h(y)$$

$$h((1-\lambda)x+\lambda y) \le (1-\lambda)h(x) + \lambda h(y)$$

It follows that h is convex.

Exercise 3

a) Assume that no x as claimed by the stated theorem exists.

That x is a probability vector is the same as $1^T x = 1$, $x \geq 0$

$Px = x$ is the same as $(P-I)x = 0$

Both $1^T x = 1$ and $(P-I)x = 0$ is the same as

$$\begin{pmatrix} P-I \\ 1^T \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If no such x exists, this system is infeasible.

b) We apply Farkas Lemma with $A = \begin{pmatrix} P-I \\ 1^T \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

no such x exists : there exists a y so that $y^T b < 0$, $y^T A \geq 0$

write $y = \begin{pmatrix} z \\ w \end{pmatrix}$ (z ← vector, w ← scalar) $\quad (y \in \mathbb{R}^{n+1},\ z \in \mathbb{R}^n,\ w \in \mathbb{R})$

$$y^T b = (z^T \ \ w)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = w < 0$$

$$y^T A = (z^T \ \ w)\begin{pmatrix} P-I \\ 1^T \end{pmatrix} = \underbrace{z^T(P-I) + w\,1^T}\ \geq 0$$

This is the same as $\quad P^T z \geq z - w1 > z \quad (\text{since } w < 0)$

It follows that $\underline{P^T z > z}$

c) Let z_j be the largest component in z.
Since P is nonnegative with columns summing to one, we get

$$(P^T z)_j = \sum_k \underbrace{p_{jk}}_{(j,k)\text{-entry in } P^T \,=\, (k,j)\text{-entry in } P} z_k \leq \sum_k p_{jk} z_j = z_j \sum_k p_{jk} = z_j$$

This contradicts $P^T z > z$ in component j, so that $P^T z > z$ does not hold.