MAT3100 - Compulsory exercise 2 of 2, 2021

April 19, 2021

Deadline

Thursday 6. may, 2021, 14:30, in Canvas (canvas.uio.no).

Instructions

You choose yourself whether to write by hand and scan your delivery, or write using a computer (for example in LaTeX). The delivery should be one PDF file. Scanned sheets should be readable.

It is expected that you present arguments for your answers that are easy to understand. Remember to include all relevant plots and figures. Students who fail on their first deilivery, but have made a real attempt to solve the exercises, will get a possibility to revise their delivery. Cooperation and all aids are permitted, but your delivery should be written by you and reflect your own understanding of the material. If we are in doubt whether you really have understood your own delivery, we may require you to explain yourself.

Application for delayed delivery

If you are ill, or for other reasons need to delay your delivery, you need to contact the study administration at the institute of mathematics (e-mail: studieinfo@math.uio.no) in good time before the deadline.

To be admitted to the final exam in this course, all compulsory exercises need to be passed in the same semester. To pass this compulsory exercise you need to make real attempts on all parts, and at least 50% needs to be answered satisfactory.

For complete guidelines for delivery of compulsory exercises, see:

www.uio.no/studier/admin/obligatoriske-aktiviteter/mn-math-oblig.html

Good luck!

Exercise 1

a)

Write down the dual problems for the following LP problems (as this is defined in section 5.2):

$$\begin{array}{llllllll} \min & 3x_1 & + & 5x_2 & - & x_3 & & \\ \text{subject to} & x_1 & - & x_2 & + & x_3 & \leq & 3 \\ & 2x_1 & - & 3x_2 & & & \leq & 4 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array} \tag{1}$$

$$\begin{array}{llllll} \max & 3x_1 & + & 2x_2 & & \\ \text{subject to} & 4x_1 & + & 2x_2 & \leq & 16 \\ & x_1 & + & 2x_2 & \leq & 8 \\ & x_1 & + & x_2 & \leq & 5 \\ & & & x_1, x_2 & \geq & 0 \end{array} \tag{2}$$

b)

Write (1) and its dual problem in matrix form. Do the same for (2).

c)

Show that $x^* = (3, 2)$ is feasible for the primal problem (2) and $y^* = (1/2, 0, 1)$ is feasible for the corresponding dual problem. Moreover, show that x^* is in fact the optimal solution of (2).

Exercise 2

Assume that $f(x)$ and $g(x)$ are convex functions defined on $\mathbb{R}^n$. Prove that $h(x) = \max(f(x), g(x))$ also is a convex function.

Exercise 3

In this exercise we will prove the following theorem:

Let P be an $n \times n$-matrix with nonnegative entries, and assume that any column sums to one (i.e., all columns in P are probability vectors). Then there exists a probability vector $\mathbf{x}$ so that $P\mathbf{x} = \mathbf{x}$.

A matrix P as above is also called a *stochastic matrix*. Stochastic matrices and probability vectors are the basis for the study of *Markov chains*: The result states that any Markov chain has an *equilibrium* (here denoted $\mathbf{x}$).

a)

Assume that no $\mathbf{x}$ as claimed in the stated theorem exists. Prove that this is equivalent to that the problem

$$\begin{pmatrix} P - I \\ \mathbf{1}^T \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}, \ \mathbf{x} \geq 0$$

is infeasible. $\mathbf{1}$ is here the column vector consisting of all ones.

b)

In exercise 17 in "a mini-introduction to convexity" the following version of Farkas lemma was proved:

$A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, *is feasible if and only if* $\mathbf{y}^T\mathbf{b} \geq \mathbf{0}$ *for all* $\mathbf{y}$ *with* $\mathbf{y}^T A \geq \mathbf{0}$.

Apply Farkas lemma to the system stated in **a)** to show that, if no $\mathbf{x}$ as claimed in the stated theorem exists, then there exists a vector $\mathbf{z} \in \mathbb{R}^n$ so that $P^T\mathbf{z} > \mathbf{z}$.

c)

Explain that, for any P with nonnegative entries and with columns summing to one, it is impossible that $P^T\mathbf{z} > \mathbf{z}$. The deduction in **b)** thus produces a contradiction, and it follows that there exists an $\mathbf{x}$ as claimed.