MAT3100 - Compulsory exercise 2 of 2, 2021

May 21, 2021

Deadline

Thursday 6. may, 2021, 14:30, in Canvas (canvas.uio.no).

Instructions

You choose yourself whether to write by hand and scan your delivery, or write using a computer (for example in LaTeX). The delivery should be one PDF file. Scanned sheets should be readable.

It is expected that you present arguments for your answers that are easy to understand. Remember to include all relevant plots and figures. Students who fail on their first deilivery, but have made a real attempt to solve the exercises, will get a possibility to revise their delivery. Cooperation and all aids are permitted, but your delivery should be written by you and reflect your own understanding of the material. If we are in doubt whether you really have understood your own delivery, we may require you to explain yourself.

Application for delayed delivery

If you are ill, or for other reasons need to delay your delivery, you need to contact the study administration at the institute of mathematics (e-mail: studieinfo@math.uio.no) in good time before the deadline.

To be admitted to the final exam in this course, all compulsory exercises need to be passed in the same semester. To pass this compulsory exercise you need to make real attempts on all parts, and at least 50% needs to be answered satisfactory.

For complete guidelines for delivery of compulsory exercises, see:

www.uio.no/studier/admin/obligatoriske-aktiviteter/mn-math-oblig.html

Good luck!

Exercise 1

a)

Write down the dual problems for the following LP problems (as this is defined in section 5.2):

$$\begin{array}{lrcrcrcl} \text{min} & 3x_1 & + & 5x_2 & - & x_3 & & \\ \text{subject to} & x_1 & - & x_2 & + & x_3 & \leq & 3 \\ & 2x_1 & - & 3x_2 & & & \leq & 4 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array} \tag{1}$$

$$\begin{array}{lrcrcl} \text{max} & 3x_1 & + & 2x_2 & & \\ \text{subject to} & 4x_1 & + & 2x_2 & \leq & 16 \\ & x_1 & + & 2x_2 & \leq & 8 \\ & x_1 & + & x_2 & \leq & 5 \\ & & & x_1, x_2 & \geq & 0 \end{array} \tag{2}$$

Solution: For (1), the expression is a bit non-standard when compared to what we are used to: In a minimization problem we are used to having $\geq$-inequalities. I have not been strict if you haven't noticed this. To get into the standard form we are used to we can first rewrite to

$$\begin{array}{lrcrcrcl} \text{max} & -3x_1 & - & 5x_2 & + & x_3 & & \\ \text{subject to} & x_1 & - & x_2 & + & x_3 & \leq & 3 \\ & 2x_1 & - & 3x_2 & & & \leq & 4 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

for which the dual LP is

$$\begin{array}{lrcrcr} \text{min} & 3y_1 & + & 4y_2 & & \\ \text{subject to} & y_1 & + & 2y_2 & \geq & -3 \\ & -y_1 & - & 3y_2 & \geq & -5 \\ & y_1 & & & \geq & 1 \\ & & & y_1, y_2 & \geq & 0 \end{array}$$

which can also be written as

$$\begin{array}{lrcrcr} \text{max} & -3y_1 & - & 4y_2 & & \\ \text{subject to} & y_1 & + & 2y_2 & \geq & -3 \\ & -y_1 & - & 3y_2 & \geq & -5 \\ & y_1 & & & \geq & 1 \\ & & & y_1, y_2 & \geq & 0 \end{array}$$

For (2) the dual LP is

$$\begin{array}{lrcrcrcl} \text{min} & 16y_1 & + & 8y_2 & + & 5y_3 & & \\ \text{subject to} & 4y_1 & + & y_2 & + & y_3 & \geq & 3 \\ & 2y_1 & + & 2y_2 & + & y_3 & \geq & 2 \\ & & & & & y_1, y_2, y_3 & \geq & 0 \end{array}$$

b)

Write (1) and its dual problem in matrix form. Do the same for (2).
Solution: The general form for the primal problem is

$$\begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{subject to} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

For (1) we have that

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & 0 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

and for (2) we have that

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 16 \\ 8 \\ 5 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

To obtain the matrix form of the dual problems, we insert this into the general form of the dual problem, which is

$$\begin{array}{lrcl} \min & \mathbf{b}^T\mathbf{y} & & \\ \text{subject to} & A^T\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & \mathbf{0} \end{array}$$

c)

Show that $x^* = (3, 2)$ is feasible for the primal problem (2) and $y^* = (1/2, 0, 1)$ is feasible for the corresponding dual problem. Moreover, show that x^* is in fact the optimal solution of (2).
Solution: Inserting $x^* = (3, 2)$ in the left hand side in the constraints in (2) we get 16, 7, and 5. x^* is therefore feasible, and the slacks are 0, 1, and 0. Inserting $y^* = (1/2, 0, 1)$ in the left hand side in the constraints in the dual problem of (2), we get 3 and 2. y^* is therefore feasible, and the slacks are 0 and 0. Complmenentary slack is easily verified, so that x^* is optimal for (2). One can actually avoid computing the slacks altogether by observing that the primal objective function computes to 13 at x^*, while the dual objective function computes to 13 at y^* as well. Since the two values were equal, optimality follows from the weak duality theorem.

Exercise 2

Assume that $f(x)$ and $g(x)$ are convex functions defined on $\mathbb{R}^n$. Prove that $h(x) = \max(f(x), g(x))$ also is a convex function.
Solution: Since $f(x) \leq h(x)$ and $g(x) \leq h(x)$, and using the definition of convexity, we get

$$\begin{aligned} f((1-\lambda)x + \lambda y) &\leq (1-\lambda)f(x) + \lambda f(y) \leq (1-\lambda)h(x) + \lambda h(y) \\ g((1-\lambda)x + \lambda y) &\leq (1-\lambda)g(x) + \lambda g(y) \leq (1-\lambda)h(x) + \lambda h(y). \end{aligned}$$

From this it follows that

$$h((1-\lambda)x+\lambda y) = \max(f((1-\lambda)x+\lambda y), g((1-\lambda)x+\lambda y))) \leq (1-\lambda)h(x)+\lambda h(y),$$

so that $h(x)$ is also convex.

Exercise 3

In this exercise we will prove the following theorem:

Let P be an $n \times n$-matrix with nonnegative entries, and assume that any column sums to one (i.e., all columns in P are probability vectors). Then there exists a probability vector $\mathbf{x}$ so that $P\mathbf{x} = \mathbf{x}$.

A matrix P as above is also called a *stochastic matrix*. Stochastic matrices and probability vectors are the basis for the study of *Markov chains*: The result states that any Markov chain has an *equilibrium* (here denoted $\mathbf{x}$).

a)

Assume that no $\mathbf{x}$ as claimed in the stated theorem exists. Prove that this is equivalent to that the problem

$$\begin{pmatrix} P-I \\ \mathbf{1}^T \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}, \ \mathbf{x} \geq 0$$

is infeasible. $\mathbf{1}$ is here the column vector consisting of all ones.

Solution: That $\mathbf{x}$ is a probability vector is the same as $\mathbf{1}^T\mathbf{x} = 1$, $\mathbf{x} \geq \mathbf{0}$. That $P\mathbf{x} = \mathbf{x}$ is the same as $(P-I)\mathbf{x} = \mathbf{0}$. Both these are satisfied if and only if $\begin{pmatrix} P-I \\ \mathbf{1}^T \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$, $\mathbf{x} \geq 0$. That no such $\mathbf{x}$ exists is thus the same as this problem being infeasible.

b)

In exercise 17 in "a mini-introduction to convexity" the following version of Farkas lemma was proved:

$A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, *is feasible if and only if* $\mathbf{y}^T\mathbf{b} \geq \mathbf{0}$ *for all* $\mathbf{y}$ *with* $\mathbf{y}^T A \geq \mathbf{0}$.

Apply Farkas lemma to the system stated in **a)** to show that, if no $\mathbf{x}$ as claimed in the stated theorem exists, then there exists a vector $\mathbf{z} \in \mathbb{R}^n$ so that $P^T\mathbf{z} > \mathbf{z}$.

Solution: According to **a)**, no such $\mathbf{x}$ existing is the same as that there exists a $\mathbf{y}$ so that $\mathbf{y}^T\mathbf{b} < \mathbf{0}$ and $\mathbf{y}^T A \geq \mathbf{0}$. Setting $A = \begin{pmatrix} P-I \\ \mathbf{1}^T \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$ (so that $\mathbf{y}$ must be in $\mathbb{R}^{n+1}$), and writing $\mathbf{y} = \begin{pmatrix} \mathbf{z} \\ w \end{pmatrix}$ with $\mathbf{z} \in \mathbb{R}^n$, $w \in \mathbb{R}$, this means that

$$\mathbf{y}^T\mathbf{b} = \begin{pmatrix} \mathbf{z}^T & w \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = w < \mathbf{0}$$

and

$$\mathbf{y}^T A = \begin{pmatrix} \mathbf{z}^T & w \end{pmatrix} \begin{pmatrix} P-I \\ \mathbf{1}^T \end{pmatrix} = \mathbf{z}^T(P-I) + w\mathbf{1}^T \geq \mathbf{0}.$$

The latter is the same as $P^T\mathbf{z} \geq \mathbf{z} - w\mathbf{1} > \mathbf{z}$, where we used that $w < 0$. It follows that $P^T\mathbf{z} > \mathbf{z}$.

c)

Explain that, for any P with nonnegative entries and with columns summing to one, it is impossible that $P^T\mathbf{z} > \mathbf{z}$. The deduction in **b)** thus produces a contradiction, and it follows that there exists an $\mathbf{x}$ as claimed.

Solution: Let z_j is the largest component in $\mathbf{z}$. Since P is nonnegative with columns summing to one we get that

$$(P^T\mathbf{z})_j = \sum_k p_{jk} z_k \leq \sum_k p_{jk} z_j = z_j,$$

This contradicts $P^T\mathbf{z} > \mathbf{z}$ in component j, so that $P^T\mathbf{z} > \mathbf{z}$ does not hold.