Oblig 1

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Consider the problem

$$\begin{array}{rcccccll} \max x_1 & + & x_2 & + & 2x_3 & & \\ \text{s.t. } x_1 & + & x_2 & + & x_3 & \leq & 5 \\ x_1 & & & & & \leq & 2 \\ & & x_2 & & & \leq & 2 \\ & & & & x_3 & \leq & 2 \\ x_1, & & x_2, & & x_3 & \geq & 0 \end{array} \tag{1}$$

a)

Write down a 4×3-matrix A and vectors $\mathbf{b}$ and $\mathbf{c}$ so that (1) can be written on the form

$$\begin{aligned} \max\ & \mathbf{c}^T\mathbf{x} \\ \text{s.t. }\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Solution: We have that

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 2 \\ 2 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

b)

Solve the problem using the Simplex method and Blands rule. Is the optimal solution unique? If not, write down the general optimal solution.
Solution: The initial dictionary is

$$\begin{array}{rcrcrcrcr} \eta & = & 0 & + & x_1 & + & x_2 & + & 2x_3 \\ \hline w_1 & = & 5 & - & x_1 & - & x_2 & - & x_3 \\ w_2 & = & 2 & - & x_1 & & & & \\ w_3 & = & 2 & & & - & x_2 & & \\ w_4 & = & 2 & & & & & - & x_3 \end{array}$$

This is feasible. Blands rule would choose x_1 as entering in the first iteration, and have w_2 leave in the first iteration, to arrive at

$$\begin{array}{rclclclcl}
\eta & = & 2 & - & w_2 & + & x_2 & + & 2x_3 \\
\hline
w_1 & = & 3 & + & w_2 & - & x_2 & - & x_3 \\
x_1 & = & 2 & - & w_2 & & & & \\
w_3 & = & 2 & & & - & x_2 & & \\
w_4 & = & 2 & & & & & - & x_3
\end{array}$$

x_2 then enters in the next iteration according to Blands rule, and w_3 would leave, and we obtain

$$\begin{array}{rclclclcl}
\eta & = & 4 & - & w_2 & - & w_3 & + & 2x_3 \\
\hline
w_1 & = & 1 & + & w_2 & + & w_3 & - & x_3 \\
x_1 & = & 2 & - & w_2 & & & & \\
x_2 & = & 2 & & & - & w_3 & & \\
w_4 & = & 2 & & & & & - & x_3
\end{array}$$

Now x_3 would enter, and w_1 would leave, giving us

$$\begin{array}{rclclclcl}
\eta & = & 6 & + & w_2 & + & w_3 & - & 2w_1 \\
\hline
x_3 & = & 1 & + & w_2 & + & w_3 & - & w_1 \\
x_1 & = & 2 & - & w_2 & & & & \\
x_2 & = & 2 & & & - & w_3 & & \\
w_4 & = & 1 & - & w_2 & - & w_3 & + & w_1
\end{array}$$

The corresponding basic feasible solution is $(2, 2, 1)$. Blands rule would now choose w_2 as leaving, w_4 as entering, giving

$$\begin{array}{rclclclcl}
\eta & = & 7 & - & w_4 & & & - & w_1 \\
\hline
x_3 & = & 2 & - & w_4 & & & & \\
x_1 & = & 1 & + & w_4 & + & w_3 & - & w_1 \\
x_2 & = & 2 & & & - & w_3 & & \\
w_2 & = & 1 & - & w_4 & - & w_3 & + & w_1
\end{array}$$

This dictionary is optimal, with basic solution $(1, 2, 2)$. In summary. The optimal value is 7, and the optimal basic solution is $(1, 2, 2)$. This solution is not unique, however, since the coefficient of w_3 is zero in the objective. We can thus increase w_3 until one of the constraints is violated. We require that $w_3 \leq 2$, $w_3 \leq 1$ (the first and second constraints are satisfied regardless of how large w_3 is). The general optimal solution is thus $(1 + w_3, 2 - w_3, 2)$, for $0 \leq w_3 \leq 1$, so that the optimal solution is not unique.

c)

Write down all basic feasible solutions that the simplex method encountered in **b)**. If you instead applied the largest coefficient rule, what would the corresponding list of basic feasible solutions then be?
Solution: When using Blands rule 4 pivots were needed, and the basic feasible solutions were visited as follows:

$$(0, 0, 0) \to (2, 0, 0) \to (2, 2, 0) \to (2, 2, 1) \to (1, 2, 2).$$

If we instead apply the largest coefficient rule, x_3 will be the entering variable. The ratios are 1/5, 0, 0 and 1/2. The biggest is 1/2, so that w_4 is the leaving variable. Rewriting the fourth constraint as $x_3 = 2 - w_4$, and inserting this in the dictionary, we obtain

$$\begin{array}{rcrcrcrcr} \eta & = & 4 & + & x_1 & + & x_2 & - & 2w_4 \\ \hline w_1 & = & 3 & - & x_1 & - & x_2 & + & w_4 \\ w_2 & = & 2 & - & x_1 & & & & \\ w_3 & = & 2 & & & - & x_2 & & \\ x_3 & = & 2 & & & & & - & w_4 \end{array}$$

Now, letting x_1 enter (alternatively, you could also choose x_2), the ratios are 1/3, 1/2, 0, and 0. 1/2 is biggest so that w_2 leaves. We rewrite the second constraint as $x_1 = 2 - w_2$, and insert in the dictionary:

$$\begin{array}{rcrcrcrcr} \eta & = & 6 & - & w_2 & + & x_2 & - & 2w_4 \\ \hline w_1 & = & 1 & + & w_2 & - & x_2 & + & w_4 \\ x_1 & = & 2 & - & w_2 & & & & \\ w_3 & = & 2 & & & - & x_2 & & \\ x_3 & = & 2 & & & & & - & w_4 \end{array}$$

The corresponding basic feasible solution is $(2, 0, 2)$. Now x_2 is entering, and the ratios are 1, 0, 1/2, and 0. w_1 thus leaves, and by inserting $x_2 = 1 + w_2 - w_1 + w_4$ we get

$$\begin{array}{rcrcrcrcr} \eta & = & 7 & & & - & w_1 & - & w_4 \\ \hline x_2 & = & 1 & + & w_2 & - & w_1 & + & w_4 \\ x_1 & = & 2 & - & w_2 & & & & \\ w_3 & = & 1 & - & w_2 & + & w_1 & - & w_4 \\ x_3 & = & 2 & & & & & - & w_4 \end{array}$$

This dictionary is optimal. The simplex methods visited the basic feasible solutions as follows:

$$(0,0,0) \to (0,0,2) \to (2,0,2) \to (2,1,2).$$

Thus, only three pivots were needed now.

d)

Write down the dual problem of (1), as well as the dual dictionaries corresponding to the initial and optimal dictionaries from **b)**. What is the optimal solution, and is it unique?

Solution: The dual problem is to minimize $\mathbf{b}^T\mathbf{y}$ subject to the constraints $A^T\mathbf{y} \geq \mathbf{c}$, and $\mathbf{y} \geq \mathbf{0}$, with A, $\mathbf{b}$, and $\mathbf{c}$ as in **a)**. We thus get

$$\begin{array}{lrcrcrcrcr} \min & 5y_1 & + & 2y_2 & + & 2y_3 & + & 2y_4 & & \\ \text{s.t.} & y_1 & + & y_2 & & & & & \geq & 1 \\ & y_1 & & & + & y_3 & & & \geq & 1 \\ & y_1 & & & & & + & y_4 & \geq & 2 \\ & y_1, & & y_2, & & y_3, & & y_4 & \geq & 0 \end{array} \tag{2}$$

The initial dual dictionary is (take the negative transpose)

$$\begin{array}{rcrcrcrcrcr} -\xi & = & & - & 5y_1 & - & 2y_2 & - & 2y_3 & - & 2y_4 \\ \hline z_1 & = & -1 & + & y_1 & + & y_2 & & & & \\ z_2 & = & -1 & + & y_1 & & & + & y_3 & & \\ z_3 & = & -2 & + & y_1 & & & & & + & y_4 \end{array}$$

The optimal dual dictionary is

$$\begin{array}{rcrcrcrcrcr} -\xi & = & -7 & - & 2z_3 & - & z_1 & - & 2z_2 & - & y_2 \\ \hline y_4 & = & 1 & + & z_3 & - & z_1 & & & + & y_2 \\ y_3 & = & & & & & -z_1 & + & z_2 & + & y_2 \\ y_1 & = & 1 & & & & z_1 & & & - & y_2 \end{array}$$

We see that the optimal dual solution is $(1, 0, 0, 1)$, and this is unique since no coefficients in the objective are 0.

e)

State what is meant by complementary slackness, and verify that this property holds for the optimal solutions you have found for the primal and dual problems in **b)** and **d)**.
Solution: In **b)** we found $(x_1, x_2, x_3) = (1, 2, 2)$, $(w_1, w_2, w_3, w_4) = (0, 1, 0, 0)$. In **e)** we found $(y_1, y_2, y_3, y_4) = (1, 0, 0, 1)$, $(z_1, z_2, z_3) = (0, 0, 0)$. It follows that

$$\mathbf{x} \cdot \mathbf{z} = (1, 2, 2) \cdot (0, 0, 0) = (0, 0, 0).$$

$$\mathbf{y} \cdot \mathbf{w} = (1, 0, 0, 1) \cdot (0, 1, 0, 0) = (0, 0, 0, 0),$$

and we have verified complementary slackness.

f)

We consider the same constraints as in **a)**, but change the objective function to $x_1 + 2x_2 + 2x_3$. Find an optimal solution to this modified problem. Is the optimal solution unique now?
Solution: Consider the optimal dictionary from **b)**. The new objective can be written as

$$x_1 + 2x_2 + 2x_3 = 1 + w_4 + w_3 - w_1 + 2(2 - w_3) + 2(2 - w_4) = 9 - w_4 - w_3 - w_1.$$

It follows that the same basis is optimal for the new objective function as well, and that the optimum $(1, 2, 2)$ is unique.