Oblig 2

Øyvind Ryan

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Problem 1 (Game theory)

For this first problem you may need to consult the lecture notes on game theory from 31. march (see the schedule of the course), as the material on pure minmax- and maxmin strategies can not be found in the book. We consider the matrix game with payoff matrix

$$A = \begin{pmatrix} -2 & -2 & 2 \\ -2 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

a)

Find the best pure strategies for the row- and column player (i.e., the minmax and maxmin strategies), and their corresponding payoffs (V^* and V_*). Show that the game does not have a value (i.e., no pure maxmin- and minmax strategies).
Solution: If we take the maxima over all the rows, we get 2, 1, and 1. The minimum of these values is $V^* = 1$. This gives that the best pure strategy for the row player (the minmax strategy) is $\mathbf{e}_2$ or $\mathbf{e}_3$, with a payoff of 1.

If we take the minima over the columns we get -2, -2, and 0. The maximum of these values is $V^* = 0$. This gives that the best pure strategy for the column player (the maxmin strategy) is $\mathbf{e}_3$, with a payoff of 0. Since $V^* \neq V_*$, there are no pure minmax/maxmin strategies, i.e., no pure strategies are optimal.

b)

State the Minimax Theorem for matrix games. Explain why the pure strategies found above are not optimal, so that both players should consider using a randomized strategy instead.

c)

Let $\mathbf{x}^* = (1/4, 0, 3/4)$ and $\mathbf{y}^* = (0, 1/4, 3/4)$ be randomized strategies for the column player and row player, respectively. Prove that these are optimal strategies, and find the value of the game. Is the game fair? If not, which player wins in the long run?
Solution: We have that

$$A\mathbf{x}^* = \begin{pmatrix} -2 & -2 & 2 \\ -2 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1/4 \\ 0 \\ 3/4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1/4 \\ 1/4 \end{pmatrix},$$

and this gives

$$\min_{\mathbf{y}} \mathbf{y}^T A\mathbf{x}^* = \min_{\mathbf{y}} \mathbf{y}^T \begin{pmatrix} 1 \\ 1/4 \\ 1/4 \end{pmatrix} = 1/4,$$

where the minimum is taken over all stocastic vectors. We also get

$$(\mathbf{y}^*)^T A = \begin{pmatrix} 0 & 1/4 & 3/4 \end{pmatrix} \begin{pmatrix} -2 & -2 & 2 \\ -2 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1/4 & -3/2 & 1/4 \end{pmatrix},$$

and this gives

$$\max_{\mathbf{x}} (\mathbf{y}^*)^T A\mathbf{x} = \begin{pmatrix} 1/4 & -3/2 & 1/4 \end{pmatrix} \mathbf{x} = 1/4.$$

Since these are equal, the two strategies are optimal, and the value of the game is 1/4. Since this is nonzero, the game is not fair. Since the value of the game is positive, the column player wins in the long run.

Problem 2 (Convexity)

For this second problem, you need to consult the notes "mini-introduction to convexity", found on the course webpage.

a)

Let f be a convex function, and α a scalar. Show that the "sub-level" set $C = \{x : f(x) \leq \alpha\}$ is convex.
Solution: Let $x, y \in C$, so that $f(x) \leq \alpha$, $f(y) \leq \alpha$. Since f is convex we have that

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \leq (1-\lambda)\alpha + \lambda\alpha = \alpha.$$

It follows that $(1-\lambda)x + \lambda y \in C$ as well, so that C is convex.

b)

Let f and g be convex functions. Show that $h(x) = \max(f(x), g(x))$ also is a convex function.
Solution: Since f and g are convex we have

$$\begin{aligned} f((1-\lambda)x + \lambda y) &\leq (1-\lambda)f(x) + \lambda f(y) \leq (1-\lambda)h(x) + \lambda h(y) \\ g((1-\lambda)x + \lambda y) &\leq (1-\lambda)g(x) + \lambda g(y) \leq (1-\lambda)h(x) + \lambda h(y). \end{aligned}$$

From this it follows that

$$h((1-\lambda)x+\lambda y) = \max(f((1-\lambda)x+\lambda y), g((1-\lambda)x+\lambda y)) \leq (1-\lambda)h(x)+\lambda h(y)$$

as well, so that $h(x)$ is convex.

c)

Show that

$$\text{conv}\{\mathbf{0}, \mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2, ..., \mathbf{e}_n, -\mathbf{e}_n\} = \{\mathbf{x} \in \mathbf{R}^n : \sum_{j=1}^{n} |x_j| \leq 1\}$$

(the left hand side is the convex hull of $2n+1$ vectors in $\mathbf{R}^n$. The $\mathbf{e}_i$ are the standard unit basis vectors in $\mathbf{R}^n$))

Solution: Denote the left hand side by P, the right hand side by Q. Q is clearly a polyhedron which contains $\mathbf{0}$, $\mathbf{e}_1, -\mathbf{e}_1$, $\mathbf{e}_2, -\mathbf{e}_2$,...,$\mathbf{e}_n, -\mathbf{e}_n$. Since a polyhedron is convex, and since the convex hull P is the smallest convex set which contains these vectors, it follows that $P \subseteq Q$.

On the other hand, assume that $\mathbf{x} \in Q$, so that $\sum_{j=1}^{n} |x_j| \leq 1$. We can write

$$\mathbf{x} = \left(1 - \sum_{j=1}^{n} |x_j|\right) \mathbf{0} + \sum_{j=1}^{n} |x_j| (\pm \mathbf{e}_j),$$

where the sign captures the sign of x_j. Since the coefficients are nonnegative and sum to one, It follows that $\mathbf{x}$ is in the convex hull P, so that also $Q \subseteq P$. Thus, we have that $P = Q$.