Some notes about duality in linear programming

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Consider the LP problem (P)

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j, \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, \ldots, m, \\ & x_j \ge 0, \quad j = 1, 2, \ldots, n. \end{aligned}$$

The dual problem (D) is

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_i y_i, \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \ge c_j, \quad j = 1, 2, \ldots, n, \\ & y_i \ge 0, \quad i = 1, 2, \ldots, m. \end{aligned}$$

The weak duality theorem says that if $x = (x_1, x_2, \ldots, x_n)$ is feasible for (P) and $y = (y_1, y_2, \ldots, y_m)$ is feasible for (D) then

$$\sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i.$$

One way to prove this is as follows. We introduce the slack variables

$$w_i := b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m,$$

$$z_j := \sum_{i=1}^{m} a_{ij} y_i - c_j, \quad j = 1, 2, \ldots, n.$$

Then

Lemma 1 (Duality)

$$\sum_{i=1}^{m} b_i y_i - \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} w_i y_i + \sum_{j=1}^{n} z_j x_j. \tag{1}$$

Proof.

$$\begin{aligned}\sum_{i=1}^{m} b_i y_i - \sum_{j=1}^{n} c_j x_j &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j + w_i \right) y_i - \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i - z_j \right) x_j \\ &= \sum_{i=1}^{m} w_i y_i + \sum_{j=1}^{n} z_j x_j.\end{aligned}$$

□

The weak duality theorem now follows from identity (1): for feasible x and y all the variables x_j, y_i, z_j, w_i are non-negative and so the right hand side of (1) is non-negative.

The strong duality theorem says that if (P) has an optimal solution x^* then (D) has an optimal solution y^* and

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*. \tag{2}$$

This can be proved using the simplex algorithm and the negative transpose property of the dictionaries.

The complementary slackness theorem says that feasible x and y are optimal if and only if

$$\begin{aligned} x_j z_j = 0, &\quad j = 1, 2, \ldots, n, \\ y_i w_i = 0, &\quad i = 1, 2, \ldots, m. \end{aligned} \tag{3}$$

This theorem follows from the identity (1) and the strong duality theorem. Suppose x and y are optimal. Then by (2), the left hand side of equation (1) is zero. Then the sum of all the products $x_j z_j$ and $y_i w_i$ is zero. Since these products are non-negative they must therefore be zero.

Conversely, suppose that (3) holds. Then the right hand side of equation (1) is zero, and we obtain (2).

Free variables / equality constraints.

Another kind of LP problem is (P):

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j, \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, \ldots, m. \end{aligned}$$

Here the variables $x_1, x_2, \ldots, x_n$ are free; they are not constrained to be non-negative. The dual problem (D) is now

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_i y_i, \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i = c_j, \quad j = 1, 2, \ldots, n, \\ & y_i \ge 0, \quad i = 1, 2, \ldots, m. \end{aligned}$$

In (D) the variables are non-negative but the constraints are equalities.

Here we introduce slack variables in (P),

$$w_i := b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m,$$

but there is no slack in the dual problem. For this problem there is an identity analogous to (1), but simpler.

Lemma 2

$$\sum_{i=1}^{m} b_i y_i - \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} w_i y_i. \tag{4}$$

Proof.

$$\begin{aligned}\sum_{i=1}^{m} b_i y_i - \sum_{j=1}^{n} c_j x_j &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j + w_i \right) y_i - \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j \\ &= \sum_{i=1}^{m} w_i y_i.\end{aligned}$$

□