Answers to Exercises, Week 3, MAT3100, V20

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Exercises in Week 3 are: 1.2, 2.3, 2.4, 2.9, 2.10 of Vanderbei.

Exercise 1.2

There are nine variables, each variable is the number of tickets (one per passenger) we want to sell, of each of the nine types:

$$\begin{matrix} x_Y^{IN} & x_Y^{NB} & x_Y^{IB} \\ x_B^{IN} & x_B^{NB} & x_B^{IB} \\ x_M^{IN} & x_M^{NB} & x_M^{IB} \end{matrix}$$

These variables are all non-negative. The income will then be

$$\begin{aligned} \eta = {} & 300x_Y^{IN} + 160x_Y^{NB} + 360x_Y^{IB} \\ & + 220x_B^{IN} + 130x_B^{NB} + 280x_B^{IB} \\ & + 100x_M^{IN} + 80x_M^{NB} + 140x_M^{IB}, \end{aligned}$$

and the upper bounds decided on by Ivy Air are:

$$\begin{matrix} x_Y^{IN} \le 4 & x_Y^{NB} \le 8 & x_Y^{IB} \le 3 \\ x_B^{IN} \le 8 & x_B^{NB} \le 13 & x_B^{IB} \le 10 \\ x_M^{IN} \le 22 & x_M^{NB} \le 20 & x_M^{IB} \le 18. \end{matrix}$$

Since the plane holds at most 30 passengers, there are two further constraints. For the first flight, Ithaca-Newark, we require

$$\begin{aligned} & x_Y^{IN} + x_Y^{IB} \\ & + x_B^{IN} + x_B^{IB} \\ & + x_M^{IN} + x_M^{IB} \le 30, \end{aligned}$$

and for the second flight, Newark-Boston, we require

$$\begin{aligned} & x_Y^{NB} + x_Y^{IB} \\ & + x_B^{NB} + x_B^{IB} \\ & + x_M^{NB} + x_M^{IB} \leq 30. \end{aligned}$$

This is the LP problem.

We can put it into standard form by relabelling the variables

$$\begin{array}{lll} x_1 = x_Y^{IN} & x_2 = x_Y^{NB} & x_3 = x_Y^{IB} \\ x_4 = x_B^{IN} & x_5 = x_B^{NB} & x_6 = x_B^{IB} \\ x_7 = x_M^{IN} & x_8 = x_M^{NB} & x_9 = x_M^{IB}. \end{array}$$

Then the LP problem is

$$\begin{array}{rrl} \text{maximize} & c^T x & \\ \text{subject to} & Ax & \leq b, \\ & x & \geq 0, \end{array}$$

where $x = [x_1, \ldots, x_9]^T$,

$$c = [300, 160, 360, 220, 130, 280, 100, 80, 140]^T,$$

$$b = [4, 8, 3, 8, 13, 10, 22, 20, 18, 30, 30]^T,$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Using the simplex method the solution comes out to be:

$$\begin{array}{lll} x_1 = x_Y^{IN} = 4 & x_2 = x_Y^{NB} = 8 & x_3 = x_Y^{IB} = 3 \\ x_4 = x_B^{IN} = 8 & x_5 = x_B^{NB} = 9 & x_6 = x_B^{IB} = 10 \\ x_7 = x_M^{IN} = 5 & x_8 = x_M^{NB} = 0 & x_9 = x_M^{IB} = 0. \end{array}$$

Exercise 2.3

The LP problem is

$$\begin{array}{rll} \text{maximize} & 2x_1 - 6x_2 & \\ \text{subject to} & -x_1 - x_2 - x_3 & \leq -2, \\ & 2x_1 - x_2 + x_3 & \leq 1, \\ & x_1, x_2, x_3 & \geq 0. \end{array}$$

We need to use the two-phase method since one of the right hand sides is negative. So we solve the auxiliary problem,

$$\begin{array}{rll} \text{minimize} & x_0 & \\ \text{subject to} & -x_1 - x_2 - x_3 & \leq -2 + x_0, \\ & 2x_1 - x_2 + x_3 & \leq 1 + x_0, \\ & x_0, x_1, x_2, x_3 & \geq 0. \end{array}$$

We want to minimize x_0 to 0 to get a feasible solution to the original problem.

We can write the auxiliary problem as

$$\begin{array}{rll} \text{maximize} & -x_0 & \\ \text{subject to} & -x_1 - x_2 - x_3 - x_0 & \leq -2, \\ & 2x_1 - x_2 + x_3 - x_0 & \leq 1, \\ & x_0, x_1, x_2, x_3 & \geq 0. \end{array}$$

We introduce the slack variables

$$\begin{aligned} w_1 &= -2 + x_1 + x_2 + x_3 + x_0, \\ w_2 &= 1 - 2x_1 + x_2 - x_3 + x_0, \end{aligned}$$

and then the initial dictionary is

$$\begin{array}{rcrcrcrcrcr} \eta & = & & & & & & & & - & x_0 \\ \hline w_1 & = & -2 & + & x_1 & + & x_2 & + & x_3 & + & x_0 \\ w_2 & = & 1 & - & 2x_1 & + & x_2 & - & x_3 & + & x_0 \end{array}$$

This is not a feasible dictionary because there are negative values in the first column. However, after one iteration we will obtain a feasible dictionary. To do this we put x_0 into the basis and we take the variable which has the most negative value in the first column, in this case, w_1, which has the value -2 in the first column. The new dictionary is then

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & -2 & + & x_1 & + & x_2 & + & x_3 & - & w_1 \\
\hline
x_0 & = & 2 & - & x_1 & - & x_2 & - & x_3 & + & w_1 \\
w_2 & = & 3 & - & 3x_1 & & & - & 2x_3 & + & w_1
\end{array}
$$

This is a feasible dictionary. We now continue with the simplex algorithm as normal. We can increase any of x_1, x_2, x_3 here. If we increase x_2 then x_0 will leave the basis and we get

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & & & & - & x_0 & & & & \\
\hline
x_2 & = & 2 & - & x_1 & - & x_0 & - & x_3 & + & w_1 \\
w_2 & = & 3 & - & 3x_1 & & & - & 2x_3 & + & w_1
\end{array}
$$

This is now an optimal dictionary and the objective function η has value 0 and so we have obtained a feasible solution to the orginal problem, i.e., $x_1 = 0$, $x_2 = 2$, $x_3 = 0$. We can now return to the orginal problem. We can just drop the x_0 column, and compute the original objective function in terms of the non-basic variables:

$$
\begin{aligned}
\eta &= 2x_1 - 6x_2 = 2x_1 - 6(2 - x_1 - x_3 + w_1) \\
&= -12 + 8x_1 + 6x_3 - 6w_1.
\end{aligned}
$$

Thus in Phase II we start with the feasible dictionary

$$
\begin{array}{rcrcrcrcr}
\eta & = & -12 & + & 8x_1 & + & 6x_3 & - & 6w_1 \\
\hline
x_2 & = & 2 & - & x_1 & - & x_3 & + & w_1 \\
w_2 & = & 3 & - & 3x_1 & - & 2x_3 & + & w_1
\end{array}
$$

We now apply the simplex algorithm in the usual way. First we can put x_1 into the basis and take w_2 out:

$$
\begin{array}{rcrcrcrcr}
\eta & = & -4 & - & (8/3)w_2 & + & (2/3)x_3 & - & (10/3)w_1 \\
\hline
x_2 & = & 1 & + & (1/3)w_2 & - & (1/3)x_3 & + & (2/3)w_1 \\
x_1 & = & 1 & - & (1/3)w_2 & - & (2/3)x_3 & + & (1/3)w_1
\end{array}
$$

Then x_3 in and x_1 out:

$$
\begin{array}{rcrcrcrcr}
\eta & = & -3 & - & 3w_2 & - & x_1 & - & 3w_1 \\
\hline
x_2 & = & (1/2) & + & (1/2)w_2 & + & (1/2)x_1 & + & (1/2)w_1 \\
x_3 & = & (3/2) & - & (1/2)w_2 & - & (1/2)x_1 & + & (1/2)w_1
\end{array}
$$

Thus the solution is $x_1 = 0$, $x_2 = 1/2$, $x_3 = 3/2$ with $\eta = -3$. We also have $w_1 = w_2 = 0$.

Exercise 2.4

Again we need to use the 2-phase method. The solution is $x_1 = 0$, $x_2 = 1$, $x_3 = 0$.

Exercise 2.9

Here, we multiply the first constraint by -1 to reverse the inequality:

$$\begin{array}{rrl} \text{maximize} & 2x_1 + 3x_2 + 4x_3 & \\ \text{subject to} & 2x_1 + 3x_2 \leq 5, & \\ & x_1 + x_2 + 2x_3 & \leq 4, \\ & x_1 + 2x_2 + 3x_3 & \leq 7, \\ & x_1, x_2, x_3 & \geq 0. \end{array}$$

The solution is $x_1 = 3/2$, $x_2 = 5/2$, $x_3 = 0$ and $\eta = 21/2$.

Exercise 2.10

Here, we replace the equality constraint by two inequalities:

$$\begin{array}{rrl} \text{maximize} & 6x_1 + 8x_2 + 5x_3 + 9x_4 & \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 & \leq 1, \\ & -x_1 - x_2 - x_3 - x_4 & \leq -1, \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array}$$

Here, we need the 2-phase method. We solve the auxiliary problem:

$$\begin{array}{rrl} \text{maximize} & -x_0 & \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 - x_0 & \leq 1, \\ & -x_1 - x_2 - x_3 - x_4 - x_0 & \leq -1, \\ & x_0, x_1, x_2, x_3, x_4 & \geq 0. \end{array}$$

We introduce two slack variables and the initial dictionary is

$$\begin{array}{rcrcrcrcrcrcr} \eta & = & & & & & & & & & & - & x_0 \\ \hline w_1 & = & 1 & - & x_1 & - & x_2 & - & x_3 & - & x_4 & + & x_0 \\ w_2 & = & -1 & + & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_0 \end{array}$$

Then x_0 enters and w_2 leaves:

$$\begin{array}{rcrcrcrcrcrcr} \eta & = & -1 & + & x_1 & + & x_2 & + & x_3 & + & x_4 & - & w_2 \\ \hline w_1 & = & 2 & - & 2x_1 & - & 2x_1 & - & 2x_1 & - & 2x_1 & + & w_2 \\ x_0 & = & 1 & - & x_1 & - & x_2 & - & x_3 & - & x_4 & + & w_2 \end{array}$$

We now have a feasible dictionary for the Phase I problem. η has the value -1. We hope to increase this to 0. We can let x_1 enter. We can also let x_0 leave (since this might give us the optimal solution to Phase I):

$$
\begin{array}{rcrcrcrcrcrcr}
\eta & = & & - & x_0 & & & & & & & & \\
\hline
w_1 & = & 0 & + & 2x_0 & & & & & & & - & w_2 \\
x_1 & = & 1 & - & x_0 & - & x_2 & - & x_3 & - & x_4 & + & w_2
\end{array}
$$

And sure enough, this is the optimal solution to Phase I and since we now have $\eta = 0$, this gives us a feasible solution to the original problem. We throw away the x_0 terms and we express the original objective function in terms of the current non-basic variables, which are x_1, x_2, x_3, w_2:

$$
\begin{aligned}
\eta &= 6x_1 + 8x_2 + 5x_3 + 9x_4 \\
&= 6(1 - x_2 - x_3 - x_4 + w_2) + 8x_2 + 5x_3 + 9x_4 \\
&= 6 + 2x_2 - x_3 + 3x_4 + 6w_2.
\end{aligned}
$$

Then a feasible dictionary for Phase II is

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & 6 & + & 2x_2 & - & x_3 & + & 3x_4 & + & 6w_2 \\
\hline
w_1 & = & 0 & & & & & & & - & w_2 \\
x_1 & = & 1 & - & x_2 & - & x_3 & - & x_4 & + & w_2
\end{array}
$$

Note that this dictionary is degenerate since the basic variable w_1 has the value 0. So, if we put w_2 into the basis, we take w_1 out and the value of η will not change. We get:

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & 6 & + & 2x_2 & - & x_3 & + & 3x_4 & - & 6w_1 \\
\hline
w_2 & = & 0 & & & & & & & - & w_1 \\
x_1 & = & 1 & - & x_2 & - & x_3 & - & x_4 & - & w_1
\end{array}
$$

The variable with the largest positive coefficient is now x_4. So we now put x_4 in and take x_1 out:

$$
\begin{array}{rcrcrcrcrcr}
\eta & = & 9 & - & x_2 & - & 4x_3 & - & 3x_1 & - & 9w_1 \\
\hline
w_2 & = & 0 & & & & & & & - & w_1 \\
x_4 & = & 1 & - & x_2 & - & x_3 & - & x_1 & - & w_1
\end{array}
$$

We have reached the optimal solution: $x_1 = x_2 = x_3 = 0$, $x_4 = 1$, $\eta = 9$. We have $w_1 = 0$ since it is a non-basic variable, but we also have $w_2 = 0$ since the dictionary is degenerate (but optimal).