UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	MAT3100 — Linear optimization
Day of examination:	07 June 2024
Examination hours:	1500 – 1900

This exam paper consists of 6 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the exam paper is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 Simplex method

Consider the LP problem (P)

$$\begin{array}{rrl} \text{maximize} & x_1 + 2x_2 & \\ \text{subject to} & 3x_1 + x_2 & \leq 9, \\ & -x_1 + x_2 & \leq 1, \\ & x_1, x_2 & \geq 0. \end{array}$$

1a

Use the simplex method to find an optimal solution and the corresponding objective value.

Answer: We introduce the slack variables $w_1 = 9 - 3x_1 - x_2$ and $w_2 = 1 + x_1 - x_2$, and then the initial dictionary is

$$\begin{array}{ccccccc} \eta & = & & & x_1 & + & 2x_2 \\ \hline w_1 & = & 9 & - & 3x_1 & - & x_2 \\ w_2 & = & 1 & + & x_1 & - & x_2 \end{array}$$

We can let x_2 go into the basis. When increasing x_2, we need $x_2 \leq 9$ to keep $w_1 \geq 0$, and $x_2 \leq 1$ to keep $w_2 \geq 0$, and so w_2 leaves the basis. Then, we express

$$x_2 = 1 + x_1 - w_2,$$

and with this substitution, the new dictionary is

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$$\begin{array}{rcrcrcr} \eta & = & 2 & + & 3x_1 & - & 2w_2 \\ \hline w_1 & = & 8 & - & 4x_1 & + & w_2 \\ x_2 & = & 1 & + & x_1 & - & w_2 \end{array}$$

Now x_1 goes into the basis, and w_1 leaves, and with the substitution,

$$x_1 = 2 - \frac{1}{4}w_1 + \frac{1}{4}w_2,$$

the new dictionary is

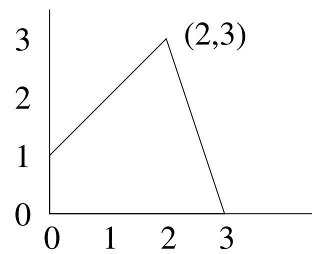
$$\begin{array}{rcrcrcr} \eta & = & 8 & - & (3/4)w_1 & - & (5/4)w_2 \\ \hline x_1 & = & 2 & - & (1/4)w_1 & + & (1/4)w_2 \\ x_2 & = & 3 & - & (1/4)w_1 & - & (3/4)w_2 \end{array}$$

This dictionary is optimal, and so an optimal solution is $x_1 = 2$, $x_2 = 3$ (and $w_1 = w_2 = 0$) and the optimal objective value is $\eta = 8$.

1b

Make a plot of the feasible region for (P) and indicate your optimal solution.

Answer:



1c

(i) Let (P′) be the LP problem formed by adding the constraint, $x_1 - 2x_2 \leq 4$, to (P). What is the optimal objective value of (P′)?
(ii) Let (P″) be the LP problem formed by replacing the objective function of (P) by $x_1 + 3x_2$. What is the optimal objective value of (P″)?

Answer: (i) Since the set of points

$$\{(x_1, x_2) : x_1 - 2x_2 \leq 4\}$$

contains the feasible region, the point $(x_1, x_2) = (2, 3)$ is also an optimal solution for (P′) and the optimal objective value of (P′) is the same as that of (P), i.e., $\eta = 8$.

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(ii) we can represent the new objective function η' in terms of w_1 and w_2:

$$\begin{aligned}\eta' &= x_1 + 3x_2 \\ &= (2 - \frac{1}{4}w_1 + \frac{1}{4}w_2) + 3(3 - \frac{1}{4}w_1 - \frac{3}{4}w_2) \\ &= 11 - w_1 - 2w_2,\end{aligned}$$

and this is the new top line of the dictionary. The coefficients of w_1 and w_2 are again negative and so the new dictionary is optimal. The solution is $(x_1, x_2) = (2, 3)$ as before and the new objective value is $\eta' = 11$.

1d

What is the dual problem (D) of (P)?

Answer:

$$\begin{array}{rrl} \text{minimize} & 9y_1 + y_2 & \\ \text{subject to} & 3y_1 - y_2 & \geq 1, \\ & y_1 + y_2 & \geq 2, \\ & y_1, y_2 & \geq 0. \end{array}$$

1e

What is an optimal solution to (D) and what is the corresponding optimal value?

Answer: The dual variables (y_1, y_2) correspond to the slack variables (w_1, w_2) and the dual slack variables (z_1, z_2) correspond to the primal variables (x_1, x_2). Using the negative-transpose property of the simplex method, the dual dictionary corresponding to the optimal primal dictionary is

$$\begin{array}{rcrcrcr} -\zeta & = & -8 & - & 2z_1 & - & 3z_2 \\ \hline y_1 & = & 3/4 & + & (1/4)z_1 & + & (1/4)z_2 \\ y_2 & = & 5/4 & - & (1/4)z_1 & + & (3/4)z_2 \end{array}$$

This dictionary is also optimal, and the optimal dual solution is $z_1 = z_2 = 0$ and $y_1 = 3/4$ and $y_2 = 5/4$, and the optimal objective value is again 8.

Problem 2 Standard form

Convert the LP problem

$$\begin{array}{rrrrl} \text{minimize} & 3x_1 & -4x_2 & -2x_3 & \\ \text{subject to} & -x_1 & +2x_2 & +3x_3 & \geq 2, \\ & 2x_1 & -x_2 & & = 5, \\ & & x_1, x_3 \geq 0 & & \end{array}$$

(Continued on page 4.)

into standard form (the form suitable for the simplex algorithm). Note that $x_2 \in \mathbb{R}$ is a free variable. What form of the simplex algorithm will be required to solve it? (do not try to solve it).

Answer: We can convert the problem as follows:

$$\begin{array}{rrrrl} \text{maximize} & -3x_1 & +4x_2 & +2x_3 & \\ \text{subject to} & x_1 & -2x_2 & -3x_3 & \leq -2, \\ & 2x_1 & -x_2 & & \leq 5, \\ & -2x_1 & +x_2 & & \leq -5, \\ & & x_1, x_3 \geq 0. & & \end{array}$$

To deal with the free variable x_2 we let $x_2 = y - z$ for $y, z \geq 0$. Then the standard form is

$$\begin{array}{rrrrrl} \text{maximize} & -3x_1 & +4y & -4z & +2x_3 & \\ \text{subject to} & x_1 & -2y & +2z & -3x_3 & \leq -2, \\ & 2x_1 & -y & +z & & \leq 5, \\ & -2x_1 & +y & -z & & \leq -5, \\ & & x_1, x_3, y, z \geq 0. & & & \end{array}$$

The 2-phase simplex method will be required because the right hand side is not non-negative.

Problem 3 Convexity

3a

Recall that a polytope P is the convex hull of a finite set of points $\mathbf{x}_1, \ldots, \mathbf{x}_t \in \mathbb{R}^n$. Show that P is convex.

Answer: Let $\mathbf{x}, \mathbf{y} \in P$ and let $\lambda \in [0,1]$. Then we need to show that $(1-\lambda)\mathbf{x} + \lambda\mathbf{y} \in P$. By definition,

$$\mathbf{x} = \sum_{i=1}^{t} \mu_i \mathbf{x}_i, \quad \mathbf{y} = \sum_{i=1}^{t} \nu_i \mathbf{x}_i,$$

for $\mu_i, \nu_i \geq 0$ and $\sum_{i=1}^t \mu_i = \sum_{i=1}^t \nu_i = 1$. Then

$$\begin{aligned} (1-\lambda)\mathbf{x} + \lambda\mathbf{y} &= (1-\lambda)\sum_{i=1}^{t} \mu_i \mathbf{x}_i + \lambda \sum_{i=1}^{t} \nu_i \mathbf{x}_i \\ &= \sum_{i=1}^{t} ((1-\lambda)\mu_i + \lambda\nu_i)\mathbf{x}_i. \end{aligned}$$

Since

$$(1-\lambda)\mu_i + \lambda\nu_i \geq 0,$$

and since

$$\sum_{i=1}^{t} ((1-\lambda)\mu_i + \lambda\nu_i) = (1-\lambda)\sum_{i=1}^{t} \mu_i + \lambda \sum_{i=1}^{t} \nu_i = (1-\lambda) + \lambda = 1,$$

it follows that $(1-\lambda)\mathbf{x} + \lambda\mathbf{y} \in P$.

(Continued on page 5.)

3b

Let A_1 and A_2 be $m \times n$ matrices and let $\mathbf{b}_1$ and $\mathbf{b}_2$ be vectors in $\mathbb{R}^m$. Show that the set

$$S := \{\mathbf{x} \in \mathbb{R}^n : A_1\mathbf{x} \le \mathbf{b}_1, A_2\mathbf{x} = \mathbf{b}_2\}$$

is convex (if it is non-empty).

Answer: Let $\mathbf{x}, \mathbf{y} \in S$ and let $\lambda \in [0,1]$. Then we need to show that $\mathbf{z} := (1-\lambda)\mathbf{x} + \lambda\mathbf{y} \in S$. By the linearity of matrix multiplication,

$$A_1\mathbf{z} = (1-\lambda)A_1\mathbf{x} + \lambda A_1\mathbf{y} \le (1-\lambda)\mathbf{b}_1 + \lambda\mathbf{b}_1 = \mathbf{b}_1,$$

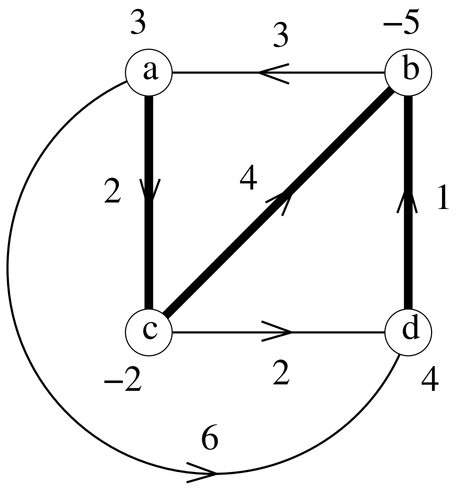
and

$$A_2\mathbf{z} = (1-\lambda)A_2\mathbf{x} + \lambda A_2\mathbf{y} = (1-\lambda)\mathbf{b}_2 + \lambda\mathbf{b}_2 = \mathbf{b}_2,$$

and therefore $\mathbf{z} \in S$.

Problem 4 Network flow

Consider the minimum cost network flow problem based on the directed graph shown in the figure. The number associated with each directed edge



(i,j) is its cost per unit flow, $c_{i,j}$, and the the number associated with each node i is its supply, b_i.

4a

Let T_1 be the spanning tree formed by the edges (a,c), (c,b), (d,b), shown in bold in the figure. Compute the tree solution x corresponding to T_1.

Answer: The flow balance equation at node i is

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{k:(k,i)\in E} x_{ki} = b_i.$$

(Continued on page 6.)

By 'tree solution' we mean that there is zero flow on edges not in T_1, i.e.,

$$x_{ba} = x_{cd} = x_{ad} = 0.$$

We use leaf elimination to find the x_{ij} in T_1:

- flow balance at a gives $x_{ac} = 3$,
- flow balance at c gives $x_{cb} - x_{ac} = -2$, and so $x_{cb} = 1$,
- flow balance at b gives $-x_{cb} - x_{db} = -5$, and so $x_{db} = 4$.

4b

Use the network simplex method to find an optimal solution and optimal value for the flow problem.

Answer: x above is non-negative and therefore a feasible solution. We compute the dual variables using $y_j = y_i + c_{ij}$ for each edge (i, j) in T_1. Use node a as the root and set $y_a = 0$. Then

$$y_c = 2, \quad y_b = 6, \quad y_d = 5.$$

We now compute the dual slacks $z_{ij} = c_{ij} - (y_j - y_i)$ on the edges (i, j) not in T_1:

$$z_{ba} = 9, \quad z_{cd} = -1, \quad z_{ad} = 1.$$

Since z_{cd} is negative, x is not an optimal solution. So, we pivot. We take x_{cd} into the basis. If we increase x_{cd} from 0 to ϵ, we obtain a loop of edges (c, d), (d, b), (c, b) with changed flow. The flows on these edges change to

$$x_{cd} = \epsilon, \quad x_{db} = 4 + \epsilon, \quad x_{cb} = 1 - \epsilon,$$

while the other flows are unchanged. The maximum allowed increase in x_{cd} is therefore $\epsilon = 1$, and this makes $x_{cb} = 0$, and so x_{cb} leaves the basis. This gives us a new spanning tree T_2 with edges (a, c), (c, d), (d, b), and the new tree solution x is given by

$$x_{ba} = x_{cb} = x_{ad} = 0.$$

and

$$x_{ac} = 3, \quad x_{cd} = 1, \quad x_{db} = 5.$$

The dual variables, with $y_a = 0$, are now

$$y_c = 2, \quad y_b = 5, \quad y_d = 4,$$

and then

$$z_{ba} = 8, \quad z_{cb} = 1, \quad z_{ad} = 2.$$

The z_{ij} are now all non-negative and so x is an optimal solution. The optimal objective value (minimum cost) is

$$\sum_{(i,j) \in T_2} c_{ij} x_{ij} = c_{ac} x_{ac} + c_{cd} x_{cd} + c_{db} x_{db} = 2 \times 3 + 2 \times 1 + 1 \times 5 = 13.$$

Good luck!

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