MAT3100, Spring 2024
Compulsory Assignment 1

Deadline 14 March, 14:30

Problem 1

1a)

Consider the LP problem

$$
\begin{array}{lrcrcrcl}
\text{maximize} & -7x_1 & & & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
& & - & 3x_2 & + & 4x_3 & \leq & 1, \\
& x_1 & - & x_2 & & & \leq & 2, \\
& -3x_1 & & & + & x_3 & \leq & 0, \\
& & & & & x_1, x_2, x_3 & \geq & 0.
\end{array} \tag{1}
$$

Identify vectors x, c, b and a matrix A such that (1) can be written

$$\max c^T x \quad \text{subject to} \quad Ax \leq b,\ x \geq 0.$$

1b)

Use the simplex algorithm to find an optimal solution.

1c)

Consider the LP problem

$$
\begin{aligned}
&\max \sum_{j=1}^{n} c_j x_j, \\
&\text{subject to} \\
&\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m, \\
&x_j \geq 0, \quad j = 1, \ldots, n,
\end{aligned} \tag{2}
$$

where c_j, a_{ij}, b_i are given numbers. Introduce slack variables $x_{n+1}, \ldots, x_{n+m}$ and identify vectors x, c, b and a matrix A such that (2) can be written

$$\max c^T x \quad \text{subject to} \quad Ax = b,\ x \geq 0. \tag{3}$$

Problem 2

Consider the LP problem

$$
\begin{array}{lrcrcl}
\text{maximize} & -3x_1 & + & 6x_2 & & \\
\text{subject to} & & & & & \\
& 2x_1 & + & x_2 & \leq & 6, \\
& -x_1 & + & 2x_2 & \leq & 2, \\
& & & x_1, x_2 & \geq & 0.
\end{array} \tag{4}
$$

2a)

Use the simplex algorithm to find all optimal solutions.

2b)

Draw the feasible region of the LP problem (4) and highlight the optimal solutions in your drawing. Explain why the non-uniqueness of optimal solutions occurs.

2c)

Use the simplex algorithm to show that the following LP problem is unbounded:

$$
\begin{array}{lrcrcl}
\text{maximize} & 3x_1 & + & 2x_2 & & \\
\text{subject to} & & & & & \\
 & x_1 & - & x_2 & \leq & 3, \\
 & x_1 & & & \leq & 2, \\
 & & & x_1, x_2 & \geq & 0.
\end{array} \tag{5}
$$

Problem 3 — Linear Regression

Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be given points in the plane, where b_i is assumed to be a function of a_i. In this exercise we will study the *linear regression* approach to finding a function of a given kind that best matches the observed points. For more details, see chapter 12 in the book.

We want to find a line $y(t) = x_1 t + x_2$ which is a best fit to the observations. This means that we want to determine the values of x_1 and x_2 that give a best possible solution to the overdetermined set of equalities

$$
\begin{aligned}
a_1 x_1 + x_2 &= b_1, \\
a_2 x_1 + x_2 &= b_2, \\
&\vdots \\
a_n x_1 + x_2 &= b_n.
\end{aligned}
$$

We define

$$
A = \begin{bmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.
$$

If $\| \cdot \|$ is some norm on $\mathbb{R}^n$ we are looking to find the vector $x \in \mathbb{R}^2$ that minimizes $\|b - Ax\|$. Recall the definitions of the 1 and 2 norms:

$$
\|z\|_1 = \sum_i |z_i|, \qquad \|z\|_2 = \sqrt{\sum_i z_i^2}.
$$

L_1 regression is regression using the 1-norm, and similarly L_2 regression is regression using the 2-norm. L_2 regression is often called the *method of least* *squares*, and it can be shown that the optimal solution to

$$\arg\min_x \|b - Ax\|_2 \tag{6}$$

is $(A^TA)^{-1}A^Tb$. For L_1 regression the problem is

$$\text{minimize } \|b - Ax\|_1 = \sum_i |b_i - \sum_j a_{ij}x_j|. \tag{7}$$

There is no simple formula for the solution to this but we can formulate the problem as a linear programming problem.

3a)

Explain why (7) can be rewritten as

$$\begin{aligned} &\text{minimize } \sum_i t_i \\ &\text{subject to } -t_i \leq b_i - \sum_j a_{ij}x_j \leq t_i, \quad i = 1, 2, \dots, n. \end{aligned} \tag{8}$$

3b)

We will investigate the difference between L_1 and L_2 regression for the following 10 data points (i.e., $n = 10$):

$$(0, 2), (1, 2), (2, 4), (3, 9), (4, 6), (5, 10), (6, 11), (7, 20), (8, 16), (9, 20).$$

- Solve (6) using the formula $(A^TA)^{-1}A^Tb$.
- Solve (8) using the simplex method. You can use the matlab routine 'simplex.m' from the course webpage for this. Note: in (8) the variables are not assumed to be nonnegative.
- Plot the two straight lines along with the points.
- Compare the two straight lines. Is the difference as expected? Is there a reason to prefer one method over the other?

Your delivery should be a short report summarizing your work as a **single pdf file**, submitted through canvas.