Exercise 3.12

September 9, 2020

Let $x_i = ih$ and $x_{i+1/2} = (i+1/2)h$. We write $f_i = f(x_i)$ and $s_{i+1/2} = s(x_{i+1/2})$. We consider quadratic splines which interpolates f in x_i, $i = 0, \dots, n$, and assume the values $s(x_{i+1/2})$ for $i = 1, \dots, n-2$ are known. This gives the following.

Consider $i = 1, \dots, n-3$, and observe that on the interval $[x_i, x_{i+1}]$, we can write s as

$$s(x) = f_i + \frac{1}{h}(f_{i+1} - f_i)(x - x_i) + \frac{2}{h^2}\left(f_i - 2s_{i+1/2} + f_{i+1}\right)(x - x_i)(x - x_{i+1}).$$

This means that on $[x_i, x_{i+1}]$ we have

$$s'(x) = \frac{1}{h}(f_{i+1} - f_i) + \frac{2}{h^2}\left(f_i - 2s_{i+1/2} + f_{i+1}\right)(x - x_i) + \frac{2}{h^2}\left(f_i - 2s_{i+1/2} + f_{i+1}\right)(x - x_{i+1}).$$

Since we want s' to be continuous at each x_i, this gives

$$\begin{aligned} s'(x_{i+1}) &= \frac{1}{h}(f_{i+1} - f_i) + \frac{2}{h}\left(f_i - 2s_{i+1/2} + f_{i+1}\right) \\ &= \frac{1}{h}(f_{i+2} - f_{i+1}) - \frac{2}{h}\left(f_{i+1} - 2s_{i+3/2} + f_{i+2}\right) \end{aligned}$$

which gives the equation

$$\begin{gathered} f_{i+1} - f_i + 2f_i - 4s_{i+1/2} + 2f_{i+1} = f_{i+2} - f_{i+1} - 2f_{i+1} + 4s_{i+3/2} - 2f_{i+2} \\ 4(s_{i+1/2} + s_{i+3/2}) = f_i + 6f_{i+1} + f_{i+2} \end{gathered}$$