Exercise 3.12

September 11, 2020

Let $x_i = ih$ and $x_{i+1/2} = (i+1/2)h$ for $h > 0$. To shorten notation sometimes write $f_i = f(x_i)$ and $s_{i+1/2} = s(x_{i+1/2})$.

We consider quadratic splines which interpolates f at x_i, $i = 0, \ldots, n$, and assume the values $s(x_{i+1/2})$ for $i = 1, \ldots, n-2$ are given. We start by arguing that this uniquely determines s on $[x_0, x_n]$. Observe that on each of the intervals $[x_0, x_{1+1/2}]$, $[x_{n-3/2}, x_n]$ and $\{[x_{i+1/2}, x_{i+3/2}]\}_{i=1}^{n-3}$, s has to interpolate three function values. Since the union of these intervals is $[x_0, x_n]$, and s is determined uniquely of each of the subintervals, this implies that s is defined uniquely on the interval $[x_0, x_n]$.

Next let $i \in \{1, \ldots, n-3\}$, and observe that on the interval $[x_{i+1/2}, x_{i+3/2}]$, s has to interpolate $s(x_{i+1/2}), s(x_{i+3/2})$ and $f(x_{i+2})$. This means that s can be expressed as

$$s(x) = s_{i+1/2} + \frac{1}{h}(s_{i+3/2} - s_{i+1/2})(x - x_{i+1/2}) + \frac{2}{h^2}\left(s_{i+1/2} - 2f_{i+1} + s_{i+3/2}\right)(x - x_{i+1/2})(x - x_{i+3/2}),$$

on $[x_{i+1/2}, x_{i+3/2}]$. Moreover, we have that

$$\begin{aligned} s'(x) = \frac{1}{h}(s_{i+3/2} - s_{i+1/2}) &+ \frac{2}{h^2}\left(s_{i+1/2} - 2f_{i+1} + s_{i+3/2}\right)(x - x_{i+1/2}) \\ &+ \frac{2}{h^2}\left(s_{i+1/2} - 2f_{i+1} + s_{i+3/2}\right)(x - x_{i+3/2}), \end{aligned}$$

on $[x_{i+1/2}, x_{i+3/2}]$.

We want s' to be continuous at all points. This means that we need to glue s' together at all the interior knots $x_{j+1/2}$, $j = 1, \ldots, n-2$. Continuity of s' at all of these points implies the following equation.

$$\begin{aligned} s'(x_{j+3/2}) &= \frac{1}{h}(s_{j+3/2} - s_{j+1/2}) + \frac{2}{h}\left(s_{j+1/2} - 2f_{j+1} + s_{j+3/2}\right) \\ &= \frac{1}{h}(s_{j+5/2} - s_{j+3/2}) - \frac{2}{h}\left(s_{j+3/2} - 2f_{j+2} + s_{j+5/2}\right). \end{aligned}$$

which gives the equation

$$s_{j+3/2} - s_{j+1/2} + 2s_{j+1/2} - 4f_{j+1} + 2s_{j+3/2} = s_{j+5/2} - s_{j+3/2} + -2s_{j+3/2} + 4f_{j+2} - 2s_{j+5/2},$$

which can be written as

$$s_{j+1/2} + 6s_{j+3/2} + s_{j+5/2} = 4(f_{j+1} + f_{j+2})$$

for $j = 1, \ldots, n-3$, as desired. In particular, if $s_{j+1/2}$ does not follow this recurrence relation, then s' will not be continuous at the interior knots.