Optimization note Ex 7.

$$f(x) = x^T A x - b^T x$$

$b, x \in \mathbb{R}^n$

$A \in \mathbb{R}^{n \times n}$

A is positive definit

We have that f is a (strictly) convex function. ~~also~~ Moreover $\nabla f(x) = Ax - b$

By Thm 4. in optimization note we know that a point $x^\circ \in \mathbb{R}^n$ is a local minimum if and only if $\nabla f(x^\circ) = 0$. That is $Ax^\circ - b = 0$

$$\Rightarrow Ax^\circ = b$$

~~Since~~ This minimum is unique since f is strictly convex. (Can also see this by observing that A is non-singular).

In the steepest decent method we choose a starting-point $x_0 \in \mathbb{R}^n$ and compute the next iterate

$$x_1 = x_0 - \alpha d_0$$

where $d_0 = \nabla f(x_0) = Ax_0 - b$ and α is chosen as the exact step length.

From the lecture notes we know that the step length with exact line search for a quadratic function is

$$\alpha = \frac{d_0^T d_0}{d_0^T A d_0}$$

Assume that $d_0 = Ax_0 - b$ is an eigenvector of A with eigenvalue λ. Then

$$\alpha = \frac{d_0^T d_0}{\lambda d_0^T d_0} = \frac{1}{\lambda}$$

This gives

$$Ax_1 = Ax_0 - \frac{1}{\lambda} A d_0 = Ax_0 - Ax_0 + b = b$$

$$\Rightarrow Ax_1 = b$$

From our observation above we then know that x_1 is a optimal point.