Gauss-Radau, with $n = 3$, $w = 1$ and $[-1, 1]$

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Let

$$I(f) = \int_{-1}^{1} f(x)\mathrm{d}x \quad \text{and} \quad I_n(f) = \sum_{i=1}^{n} w_i f(x_i).$$

Recall that with Gauss quadrature, we find weights $\{w_i\}_{i=1}^{3}$ and abscissae $\{x_i\}_{i=1}^{n}$, such that

$$I[p] = I_n[p] \quad \text{for all} \quad p \in \mathbb{P}_{2n-1}[-1, 1], \tag{1}$$

i.e., the quadrature rule is exact for polynomials of degree (at most) $2n - 1$.

In this exercise we consider $n = 3$, and fix one of the abscissae $x_3 = -1$, so that it equals one of the end points. Since we have fixed one of the abscissae, we no longer have full freedom in our choice of x_i, and we can not expect (1) to hold. Instead we will try to design a rule, such that

$$I[p] = w_1 p(x_1) + w_2 p(x_2) + w_3 p(-1) \quad \text{for all} \quad p \in \mathbb{P}_{2n-2}[-1, 1]. \tag{2}$$

That is, the quadrature rule should only be exact for polynomials of degree at most $2n - 2$.

Step 1. Observe that every $p \in \mathbb{P}_{2n-2}$ can be written as

$$p(x) = (x - (-1))q(x) + r(x)$$

where $q \in \mathbb{P}_{2n-3}$ and $r \in \mathbb{P}_0$. Moreover, we see that $p(-1) = r(-1)$, and that this uniquely determines r. This gives

$$\begin{aligned}
\int_{-1}^{1} p(x)\mathrm{d}x &= \int_{-1}^{1} (x - (-1))q(x)\mathrm{d}x + \int_{-1}^{1} r(x)\mathrm{d}x \\
&= \int_{-1}^{1} q(x)\tilde{w}(x)\mathrm{d}x + p(-1)\int_{-1}^{1} 1\mathrm{d}x \\
&= \int_{-1}^{1} q(x)\tilde{w}(x)\mathrm{d}x + 2p(-1)
\end{aligned} \tag{3}$$

where $\tilde{w}(x) = (x - (-1))$, is a non-negative weight function, and we use that $r(x) = p(-1)$ is constant.

Our next goal is the design a Gauss quadrature rule, for the weighted integral

$$\int_{-1}^{1} f(x)\tilde{w}(x)\mathrm{d}x.$$

so that this is exact for polynomials of degree $2(n - 1) - 1 = 2n - 3 = 3$. We will do this in several steps.

Step 2. We start by finding a polynomial of degree $n - 1 = 2$, which is orthogonal to all polynomials of degree less than $n - 1$, with respect to the inner-product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\tilde{w}(x)\mathrm{d}x.$$

We start by determining the coefficients α, c_0 and c_1, such that $\{1, x + \alpha, x^2 + c_1 x + c_0\}$ becomes an orthogonal basis for $\mathbb{P}_2$, with the given inner-product. We have

$$\langle 1, x + \alpha \rangle = \int_{-1}^{1} (x + \alpha)(x + 1)\mathrm{d}x = 0 \qquad \Longrightarrow \alpha = -\frac{1}{3}.$$

This gives the two equations

$$\begin{aligned}
\left\langle 1, x^2 + c_1 x + c_0 \right\rangle &= \int_{-1}^{1} (x^2 + c_1 x + c_0)(x + 1)\mathrm{d}x = \frac{2(1 + c_1)}{3} + 2c_0 = 0 \\
\left\langle x - \tfrac{1}{3}, x^2 + c_1 x + c_0 \right\rangle &= \int_{-1}^{1} (x - \tfrac{1}{3})(x^2 + c_1 x + c_0)(x + 1)\mathrm{d}x = \frac{2}{5} + \frac{2}{3}(c_1 + c_0 - \tfrac{1}{3}(1 + c_0)) - \tfrac{2}{3}c_0 = 0
\end{aligned}$$

We rewrite this as the system

$$\begin{aligned} 8 + 20c_1 &= 0 \\ \tfrac{4}{3} + \tfrac{7}{3}c_0 + c_1 &= 0, \end{aligned}$$

with solution $c_0 = -\frac{1}{5}$ and $c_1 = -\frac{2}{5}$. This means that the polynomial $x^2 - \frac{2}{5}x - \frac{1}{5}$ is orthogonal to all polynomials of degree 0 and 1, w.r.t. $\langle \cdot, \cdot \rangle$.

Step 3. The roots of the polynomial $x^2 - \frac{2}{5}x - \frac{1}{5}$ are

$$x = \frac{\frac{2}{5} \pm \sqrt{\frac{4}{25} + \frac{4}{5}}}{2} = \frac{1}{5}(1 \pm \sqrt{6})$$

and we denote them by $x_1 = \frac{1}{5}(1 - \sqrt{6})$ and $x_2 = \frac{1}{5}(1 + \sqrt{6})$. We will use these as our abscissae in the Gauss quadrature rule for polynomials in $\mathbb{P}_{2(n-1)-1}$.

Step 4. The weights for the quadrature rule for $\int_{-1}^{1} q(x)\tilde{w}(x)\mathrm{d}x$, are then computed as (see p. 139 in book)

$$\begin{aligned} \tilde{w}_1 &= \int_{-1}^{1} \frac{x - x_2}{x_1 - x_2}(x - (-1))\mathrm{d}x = 1 - \frac{2}{3\sqrt{6}} \\ \tilde{w}_2 &= \int_{-1}^{1} \frac{x - x_1}{x_2 - x_1}(x - (-1))\mathrm{d}x = 1 + \frac{2}{3\sqrt{6}} \end{aligned}$$

Step 5 (Conclusion). We now have that

$$\int_{-1}^{1} s(x)\tilde{w}(x)\mathrm{d}x = \tilde{w}_1 s(x_1) + \tilde{w}_2 s(x_2) \quad \text{for all } s \in \mathbb{P}_{2(n-1)-1}.$$

Using that $q \in \mathbb{P}_{2(n-1)-1}$ in (3), and that

$$q(x_i) = \frac{p(x_i) - p(-1)}{x_i - (-1)}$$

we get

$$\begin{aligned} \int_{-1}^{1} p(x)\mathrm{d}x &= \tilde{w}_1 q(x_1) + \tilde{w}_2 q(x_2) + 2p(-1) \\ &= \tilde{w}_1 \frac{p(x_1) - p(-1)}{\frac{1}{5}(1 - \sqrt{6}) + 1} + \tilde{w}_2 \frac{p(x_2) - p(-1)}{\frac{1}{5}(1 + \sqrt{6}) + 1} + 2p(-1) \\ &= \frac{\tilde{w}_1}{\frac{1}{5}(1 - \sqrt{6}) + 1} p(x_1) + \frac{\tilde{w}_2}{\frac{1}{5}(1 + \sqrt{6}) + 1} p(x_2) + \left(\frac{-\tilde{w}_1}{\frac{1}{5}(1 - \sqrt{6}) + 1} + \frac{-\tilde{w}_2}{\frac{1}{5}(1 + \sqrt{6}) + 1} + 2 \right) p(-1) \end{aligned}$$

as desired.