## Oppgave 6.1

$$L(f) = \int_a^b f(x)\,dx - (b-a) f\left(\tfrac{a+b}{2}\right)$$

$K(\theta) = L\left((x-\theta)_+^k\right)$ $\qquad \theta \in [a, b]$

Vet at $k = 1$ siden $L(x^k) = 0$ for $k = 0, 1$. $\qquad (x-\theta)_+ = \begin{cases} x-\theta & x \geq \theta \\ 0 & \text{ellers} \end{cases}$

$$K(\theta) = \int_a^b (x-\theta)_+\,dx - (b-a)\left(\tfrac{a+b}{2} - \theta\right)_+$$

$$= \int_\theta^b x-\theta\,dx - (b-a)\left(\tfrac{a+b}{2} - \theta\right)_+$$

$$= \left[\tfrac{1}{2}x^2 - \theta x\right]_\theta^b - (b-a)\left(\tfrac{a+b}{2} - \theta\right)_+$$

$$= \tfrac{1}{2}b^2 - \theta b - \tfrac{1}{2}\theta^2 + \theta^2 - (b-a)\left(\tfrac{a+b}{2} + \theta\right)_+$$

$$= \tfrac{1}{2}(b-\theta)^2 - (b-a)\left(\tfrac{a+b}{2} + \theta\right)_+$$

$$= \begin{cases} \tfrac{1}{2}(b-\theta)^2 - (b-a)\left(\tfrac{a+b}{2} + \theta\right) & \theta \in [a, \tfrac{a+b}{2}] \\ \tfrac{1}{2}(b-\theta)^2 & \theta \in [\tfrac{a+b}{2}, b] \end{cases}$$

Må sjekke at $K(\theta)$ ikke endrer fortegn.

Ser at $\tfrac{1}{2}(b-\theta)^2 \geq 0$.

Ser nå på

$$\tfrac{1}{2}(b-\theta)^2 - \tfrac{1}{2}(b-a)(a+b-2\theta) \qquad \theta \in [a, \tfrac{a+b}{2}]$$

$$\tfrac{1}{2}\left(b^2 - 2b\theta + \theta^2 - (ab + b^2 - 2\theta b - a^2 - ba - 2\theta a)\right)$$

$$\tfrac{1}{2}\left(\cancel{b^2} - \cancel{2b\theta} + \theta^2 - \cancel{ab} - \cancel{b^2} + \cancel{2\theta b} + a^2 + \cancel{ba} + 2\theta a\right)$$

$$\tfrac{1}{2}\left(a^2 + 2\theta a + \theta^2\right) = \tfrac{1}{2}(a+\theta)^2 \geq 0.$$

Feil: $\dfrac{1}{k!}\displaystyle\int_a^b K(\theta)\,d\theta\, f^{(k+1)}(\xi) \qquad \xi \in [a, b]$

$$\int_a^b K(\theta)\,d\theta = \int_a^{\frac{a+b}{2}} \tfrac{1}{2}(a+\theta)^2\,d\theta + \int_{\frac{a+b}{2}}^b \tfrac{1}{2}(b-\theta)^2\,d\theta$$

$$= \left[\tfrac{1}{6}(a+\theta)^3\right]_a^{\frac{a+b}{2}} + \left[-\tfrac{1}{6}(b-\theta)^3\right]_{\frac{a+b}{2}}^b$$

$$= \tfrac{1}{24}(b-a)^3$$

Feil: $\dfrac{1}{1!} \cdot \dfrac{1}{24}(b-a)^3 f^2(\xi)$

Oppgave 5.6 $\quad f \in C^{k+1}[a,b]$

a) Taylor med integral reminder

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a)$$
$$\underbrace{+ \dots + \frac{1}{k!}(x-a)^k f^{(k)}(a)}_{T_k f(x)} + \underbrace{\frac{1}{k!}\int_a^x (x-\theta)^k f^{(k+1)}(\theta)\,d\theta}_{R_k f(x)}$$

b) Integralet til $f$ tilnærmes numerisk slik at tilnærmingen er eksakt for polynomer av grad $k$

Vis at $\quad L(f) = \frac{1}{k!}\int_a^b K(\theta)\, f^{k+1}(\theta)\,d\theta$

Husk: $\quad K(\theta) := L\left((x-\theta)_+^k\right)$

Vi har at $\quad L(p) = 0 \quad$ for $p \in \mathbb{P}[a,b]$

Vi får

$$L(f) = L\left(T_k f(x) + R_k f(x)\right) = \underbrace{L\left(T_k f(x)\right)}_{0} + L\left(R_k f(x)\right)$$

$$L\left(R_k f(x)\right) = L\left(\frac{1}{k!}\int_a^x (x-\theta)^k f^{(k+1)}(\theta)\,d\theta\right)$$

$$= \frac{1}{k!}L\left(\int_a^b (x-\theta)_+^k f^{(k+1)}(\theta)\,d\theta\right)$$

Anta at $L$ og integraltegnet kan bytte plass.

$$= \frac{1}{k!}\int_a^b \underbrace{L\left((x-\theta)_+^k\right)}_{K(\theta)} f^{(k+1)}(\theta)\,d\theta$$

$$= \frac{1}{k!}\int_a^b K(\theta)\, f^{(k+1)}(\theta)\,d\theta. \qquad \square$$

---

$$\frac{1}{k!}L\left(\int_a^b (x-\theta)_+^k f^{(k+1)}(\theta)\,d\theta\right)$$

$$= \frac{1}{k!}\int_a^b\int_a^b (x-\theta)_+^k f^{k+1}(\theta)\,d\theta\,dx$$

$$-\left(\sum_{j=1}^n w_j \int_a^b (x_j-\theta)_+^k f^{(k+1)}(\theta)\,d\theta\right)$$

$$= \frac{1}{k!}\int_a^b\left(\int_a^b (x-\theta)_+^k f^{k+1}(\theta)\,dx - \sum_{j=1}^n w_j (x_j-\theta)_+^k f^{(k+1)}(\theta)\right)d\theta$$