Exercise 2

~~Pro~~ Let $T_n(x) = \cos(n \arccos(x))$, $n \in \mathbb{N}_0$

Prove that the T_n's are orthogonal to one another when using the inner product. $\langle f, g \rangle := \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx$.

Proof

Consider $n, m \in \mathbb{N}_0$, $n \neq m$.

Let $u = \arccos(x)$ and notice that $du = -\frac{1}{\sqrt{1-x^2}} dx$

$\arccos(-1) = \pi$

$\arccos(1) = 0$

Then

$$\langle T_n, T_m \rangle = \int_{-1}^{1} \frac{T_n(x) T_m(x)}{\sqrt{1+x^2}} dx = \int_{-1}^{1} \frac{\cos(n \arccos(x)) \cos(m \arccos(x))}{\sqrt{1+x^2}}$$

$$= -\int_{\pi}^{0} \cos(nu) \cos(mu) du$$

$$= \int_{0}^{\pi} \cos(nu) \cos(mu) du.$$

Husker at

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$
$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

$$\Rightarrow \ 2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$$

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$$\langle T_n, T_m \rangle = \int_0^{\pi} \cos(nu)\cos(mu)\,du$$

$$= \frac{1}{2}\int_0^{\pi} \cos((n+m)u) + \cos((n-m)u)\,du$$

$$= \frac{1}{2}\left[\frac{1}{n+m}\sin((n+m)u) + \frac{1}{n-m}\sin((n-m)u)\right]_0^{\pi} = 0.$$