

$$\bullet A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 7 \end{pmatrix}$$

QR - Factorization

- Gram - Schmidt A

$$A = (a_1, a_2)$$

$$q_1 = \frac{a_1}{\|a_1\|} = a_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$R_{1,1} = \|a_1\| = 1$$

$$w = a_2 - \langle q_1, a_2 \rangle q_1$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$q_2 = \frac{w}{\|w\|} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$R_{1,2} = \langle q_1, a_2 \rangle = 3$$

□

$$K_{2,2} = U^T U U = 2$$

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad R = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

• Gram-Schmidt B

$$B = (v_1, v_2) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$w = v_2 - \langle q_1, v_2 \rangle q_1 = 0$$

$$R_{1,2} = \langle q_1, v_2 \rangle = \frac{1}{\sqrt{5}} (2+8) = 2\sqrt{5}$$

Need to pick q_2 s.t. $\langle q_1, q_2 \rangle = 0$

$$\text{set } q_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad R = \sqrt{5} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

• Gram-Schmidt C

$$C = (c_1, c_2) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 7 \end{pmatrix}$$

$$q_1 = \frac{c_1}{\|c_1\|} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad R_{11} = \sqrt{10}$$

$$u = c_2 - \langle c_2, q_1 \rangle q_1$$

The numbers become very ugly,
skipping this.

• Givens rotation $A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$

$$R^{[1,2]} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \quad c^2 + s^2 = 1$$

$$A_{1,2} = c = 0 \quad \Rightarrow \quad s = \pm 1$$

$$R^{[1,2]} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad \square$$

$$\text{det } A = \begin{vmatrix} 0 & -2 \end{vmatrix} = 15$$

$$Q = (\Omega^{[1,2]})^T$$

• Given rotation $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$$(\Omega^{[1,2]} A)_{1,2} = -s + 2c = 0$$

$$\Rightarrow s = 2c \quad (\text{and we have } s^2 + c^2 = 1)$$

$$4c^2 + c^2 = 1$$

$$c = \pm \frac{1}{\sqrt{5}} \Rightarrow s = \pm \frac{2}{\sqrt{5}}$$

$$\Omega^{[1,2]} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\Omega^{[1,2]} B = \sqrt{5} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = R$$

$$Q = (\Omega^{[1,2]})^T$$

• Givens rotation $\underline{\underline{Z}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 7 \end{pmatrix}$

$$\left(\mathcal{R}^{[1,3]} \underline{\underline{Z}} \right)_{1,3} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \underline{\underline{Z}} = -s + 3c = 0$$

$$s = 3c \Rightarrow c = \frac{1}{\sqrt{10}} \quad (s^2 + c^2 = 1)$$

$$s = \frac{3}{\sqrt{10}}$$

$$\mathcal{R}^{[1,3]} \underline{\underline{Z}} = \frac{1}{\sqrt{10}} \begin{pmatrix} 10 & 23 \\ 0 & \sqrt{10} \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{R}^{[2,3]} = \begin{bmatrix} 1 & & \\ & c & s \\ & -s & c \end{bmatrix}$$

$$\left(\mathcal{R}^{[2,3]} \mathcal{R}^{[1,3]} \underline{\underline{Z}} \right)_{2,3} = -s + \frac{c}{\sqrt{10}} = 0$$

$$\Rightarrow \sqrt{10}^2 s = c \Rightarrow s = \frac{1}{\sqrt{11}}, c = \sqrt{\frac{10}{11}}$$

$$Q = (\Omega^{(1,2)} \quad \Omega^{(1,3)})$$

$$R = \begin{pmatrix} \sqrt{10} & 23 \\ 0 & \frac{\sqrt{10}}{11} + \sqrt{\frac{1}{110}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$$

$$\| \bar{c}x - b \| = \| QRx - b \| = \| Rx - Q^T b \|^2$$

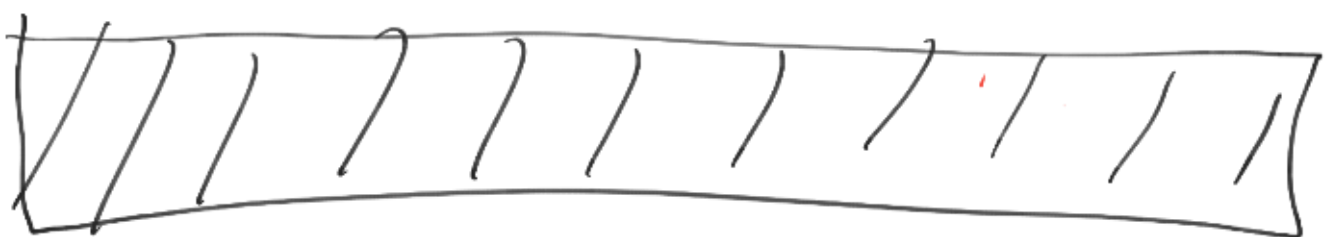
$$Q^T b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1,58 \\ -0,47 \\ -1,50 \end{pmatrix} \begin{matrix} \} b_1 \\ \} b_2 \end{matrix}$$

$$\Rightarrow \| \bar{c}x - b \| = \| R_1 x - b_1 \| + \| b_2 \|^2$$

\Rightarrow We solve $R_1 x - b_1$

to make the first term 0.

$$x = \begin{pmatrix} 1,54 \\ -0,45 \end{pmatrix}$$



1 (-4 -6)

$$A = \begin{pmatrix} 3 & -4 \\ 0 & 0 \end{pmatrix}$$

$$A = U \Sigma V^T$$

$$A^T A = V \Sigma^2 V^T$$

In Matlab

$$[\tilde{V}, \tilde{D}] = \text{eig}(A' * A)$$

$$\tilde{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \tilde{D} = \begin{pmatrix} 25 & 0 \\ 0 & 100 \end{pmatrix}$$

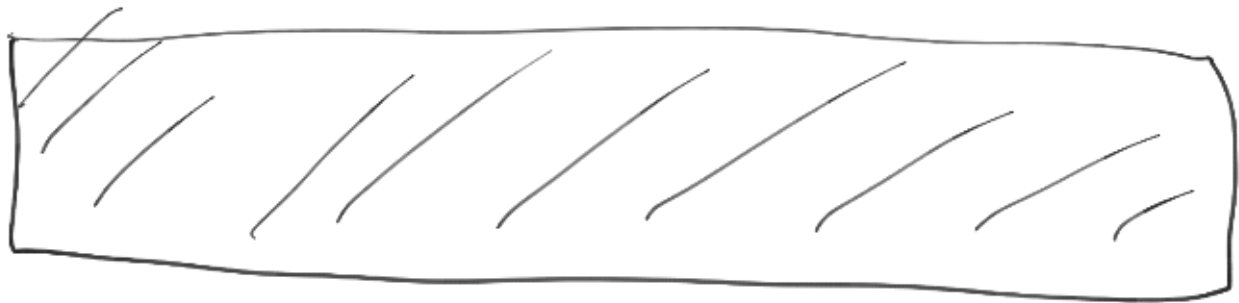
Need the right order of the singular values set $D = \begin{pmatrix} 100 & 0 \\ 0 & 25 \end{pmatrix} = \Sigma^2$

$$\text{and } V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad D^{-1/2} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$$

$$U = A V D^{-1/2} = \frac{1}{5} \begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}$$

A similar approach can

... of ... can
be used for $A \cdot A'$.



Rotation counter clockwise
in x_1, x_3 -plane by $\frac{\pi}{4}$

$$s = \sin\left(-\frac{\pi}{4}\right)$$

$$c = \cos\left(-\frac{\pi}{4}\right)$$

$$L_{1,3} = \begin{bmatrix} c & & s \\ & 1 & \\ -s & & c \end{bmatrix}$$

Sacking of components

$$D = \begin{bmatrix} 5 & & \\ & 2 & \\ & & 0 \end{bmatrix}$$

Rotation through the

reflection through the
plane orthogonal to $u = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$R = I - \frac{2uu^T}{u^T u}$$

Hence $A = R D \Omega^{[1,3]}$

We recognize R as a
Householder reflection
and $\Omega^{[1,3]}$ as a Givens
rotation. Both are orthogonal.

Set $U = R$, $\Sigma = D$

and $V = (\Omega^{[1,3]})^T$

Then $A = U \Sigma V^T$

is the singular value
decomposition of A .

