Mat 3110 - week 43-44

- $A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 7 \end{pmatrix}$

## QR - Factorization

- Gram - Schmidt $A$

$$A = (a_1 \;\; a_2)$$

$$q_1 = \frac{a_1}{\|a_1\|} = a_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$R_{1,1} = \|a_1\| = 1$$

$$w = a_2 - \langle q_1, a_2 \rangle q_1$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$q_2 = \frac{w}{\|w\|} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$R_{1,2} = \langle q_1, a_2 \rangle = 3$$

R

$R_{2,2} = \| w \| = 2$

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad R = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

- Gram-Schmidt $B$

$$B = (v_1, v_2) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$q_1 = \frac{v_1}{\| v_1 \|} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$w = v_2 - \langle q_1, v_2 \rangle q_1 = 0$$

$$R_{1,2} = \langle q_1, v_2 \rangle = \frac{1}{\sqrt{5}}\, 2 + 8 = 2\sqrt{5}$$

Need to pick $q_2$ s.t. $\langle q_1, q_2 \rangle = 0$

Set $q_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \quad R = \sqrt{5} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

- Gram-Schmidt $C$

$$C = (c_1 \;\; c_2) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 7 \end{pmatrix}$$

$$q_1 = \frac{c_1}{\|c_1\|} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \qquad R_{1,1} = \sqrt{10}$$

$$w = c_2 - \langle c_2, q_1 \rangle q_1$$

The numbers become very ugly, skipping this.

- Givens rotation $A = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$

$$G^{[1,2]} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \qquad c^2 + s^2 = 1$$

$$A_{1,2} = c = 0 \quad \Rightarrow \quad s = \pm 1$$

$G^{[1,2]}$ $\begin{bmatrix} 1 & 3 \end{bmatrix}$

$$\Omega^{[1,2]} A = \begin{bmatrix} \cdots & \cdots \\ 0 & -2 \end{bmatrix} = R$$

$$Q = \left(\Omega^{[1,2]}\right)^T$$

- Givens rotation $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$$\left(\Omega^{[1,2]} A\right)_{1,2} = -s + 2c = 0$$

$\Rightarrow s = 2c$ (and we have $s^2 + c^2 = 1$)

$$4c^2 + c^2 = 1$$

$$c = \pm \frac{1}{\sqrt{5}} \Rightarrow s = \pm \frac{2}{\sqrt{5}}$$

$$\Omega^{[1,2]} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\Omega^{[1,2]} B = \sqrt{5} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = R$$

$$Q = \left(\Omega^{[1,2]}\right)^T$$

- Givens rotation $\underline{\underline{C}} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 7 \end{pmatrix}$

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$$\left(\Omega^{[1,3]}\underline{\underline{C}}\right)_{1,3} = \begin{pmatrix} c & & s \\ & 1 & \\ -s & & c \end{pmatrix}\underline{\underline{C}} = -s+3c=0$$

$$s = 3c \quad \Rightarrow \quad c = \frac{1}{\sqrt{10}} \qquad (s^2+c^2=1)$$

$$s = \frac{3}{\sqrt{10}}$$

$$\Omega^{[1,3]}\underline{\underline{C}} = \frac{1}{\sqrt{10}}\begin{pmatrix} 10 & 23 \\ 0 & \sqrt{10} \\ 0 & 1 \end{pmatrix}$$

$$\Omega^{[2,3]} = \begin{bmatrix} 1 & & \\ & c & s \\ & -s & c \end{bmatrix}$$

$$\left(\Omega^{[2,3]}\Omega^{[1,3]}\underline{\underline{C}}\right)_{2,3} = -s + \frac{c}{\sqrt{10}} = 0$$

$$\Rightarrow \sqrt{10}\, s = c \quad \Rightarrow \quad s = \frac{1}{\sqrt{11}}, \; c = \sqrt{\frac{10}{11}}$$

$$\Omega^{\{2,3\}} \Omega^{\{1,3\}} \underline{C} = \begin{pmatrix} \sqrt{10} & \frac{23}{\sqrt{10}} \\ 0 & \sqrt{\frac{10}{11}} + \sqrt{\frac{1}{110}} \\ 0 & 0 \end{pmatrix} =: R$$

$$Q = \left( \Omega^{\{2,3\}} \Omega^{\{1,3\}} \right)^T.$$

$$\underline{C} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 7 \end{pmatrix} = (c_1, c_2)$$

$$b = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{C} x = b$ does not have a solution since $b \notin \operatorname{span}(c_1, c_2)$

Solve $\min\limits_{x \in \mathbb{R}^2} \| \underline{C} x - b \|$

$$\underline{C} = QR$$

$$Q = \left( \Omega^{[2,3]} \quad \Omega^{[1,3]} \right)$$

$$R = \begin{pmatrix} \sqrt{10} & \frac{23}{\sqrt{10}} \\ 0 & \sqrt{\frac{10}{11}} + \sqrt{\frac{1}{110}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$$

$$\| \bar{C} x - b \| = \| Q R x - b \| = \| R x - Q^T b \|$$

$$Q^T b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1{,}58 \\ -0{,}47 \\ -1{,}50 \end{pmatrix} \begin{array}{l} \} \, b_1 \\ \\ \} \, b_2 \end{array}$$

$$\Rightarrow \| \bar{C} x - b \| = \| R_1 x - b_1 \| + \| b_2 \|$$

$\Rightarrow$ We solve $R_1 x = b_1$
to make the first term 0.

$$x = \begin{pmatrix} 1{,}54 \\ -0{,}45 \end{pmatrix}$$

$1 \quad (-4 \quad -6)$

$$A = \begin{pmatrix} 3 & -8 \end{pmatrix}$$

$$A = U \Sigma V^T$$

$$A^T A = V \Sigma^2 V^T$$

In MatLab

$$[\tilde{V}, \tilde{D}] = \mathrm{eig}(A' * A)$$

$$\tilde{V} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \tilde{D} = \begin{pmatrix} 25 & 0 \\ 0 & 100 \end{pmatrix}$$

Need the rigth order of the singalar values

set $D = \begin{pmatrix} 100 & 0 \\ 0 & 25 \end{pmatrix} = \Sigma^2$

and $V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $D^{-1/2} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$

$$U = A V D^{-1/2} = \frac{1}{5} \begin{pmatrix} -3 & -4 \\ -4 & 3 \end{pmatrix}$$

A similar approach can

… or approach can
be used for $A \bullet A'$.

Rotation counter clocwise
in $x_1$-$x_3$-plane by $\frac{\pi}{4}$

$$s = \sin\left(-\frac{\pi}{4}\right)$$
$$c = \cos\left(-\frac{\pi}{4}\right)$$

$$\curvearrowleft^{[1,3]} = \begin{bmatrix} c & & s \\ 0 & 1 & \\ -s & & c \end{bmatrix}$$

Scaling of components

$$D = \begin{bmatrix} 5 & & \\ & 2 & \\ & & 0 \end{bmatrix}$$

Reflection through the

reflection through the plane orthogonal to $u = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$R = I - \frac{2uu^T}{u^Tu}$$

Hence $A = R D \Omega^{[1,3]}$

We recognize $R$ as a Householder reflection and $\Omega^{[1,3]}$ as a Givens rotation. Both are orthonormal.

Set $U = R$, $\Sigma = D$ and $V = \left(\Omega^{[1,3]}\right)^T$

Then $A = U \Sigma V^T$

is the singular value decomposition of $A$.