## Problem 1

$$\dot{x} = \lambda x \qquad x(0) = x_0$$

$$\lambda = -5 \qquad x_0 = 1$$

How large must $N$ be to guarantee linear stability of an approximation at time $T=2$ for both Eulers and implicit Eulers method?

## Euler

$$y_{n+1} = y_n + h f(y_n)$$

$$= (1 + \lambda h) y_n$$

Let $\zeta = \lambda h$. Then Eulers

method have stability function

$$R(\zeta) = 1 + \zeta$$

Stability region: $|R(\zeta)| \le 1$

$$\Rightarrow |1 - \zeta| \le 1$$

since $\lambda$ is real and $h > 0$ it implies that $\lambda h \in [-2, 0]$

$$\Rightarrow h \le \frac{2}{5}$$

We want an approximation at time $T = 2$ and we have $h = \frac{T}{N}$

This gives $h = \frac{2}{N} \le \frac{2}{5}$

$$N \geq 5$$

## Implicit Euler

$$y_{n+1} = y_n + h\, f(y_{n+1})$$

$$= y_n + \lambda h\, y_{n+1}$$

$$\Updownarrow$$

$$y_{n+1} = (1-\lambda h)^{-1} y_n$$

Stability function $R(\zeta) = (1-\zeta)^{-1}$

Stability region $|R(\zeta)| \leq 1$

Since $\zeta = \lambda h$ and $h > 0$ and

$\lambda < 0 \Rightarrow 1 - \lambda h > 1 \quad \forall\, h > 0$

$\Rightarrow (1-\lambda h)^{-1} < 1 \quad \forall\, h > 0$

Thus $N \geq 1$ is sufficient.

Thus both methods are linearly

stable if $N \geq \max\{5, 1\} = 5$.

## Problem 2

Consider $\dot{x} = Ax, \quad x(0) = x_0$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

for $\theta = -\frac{\pi}{2}$. $\Rightarrow$ $A$ is orthonormal.

$A$ has eigenvalues $\lambda_{\pm} = \pm i$

and eigenvectors $V_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$V_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\mathrm{Re}(\lambda_{\pm}) = 0 \Rightarrow$ Solution will be sinusoidal waves

Let $V = (V_+ \quad V_-)$

and $\Lambda = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$

Then $A = V \Lambda V^{-1}$

We write the system

$$\dot{x} = Ax \qquad x(0) = x_0$$

as

$$(**) \qquad \dot{u} = \Lambda u \qquad u(0) = V^{-1} x_0 = u_0$$

$$u = V^{-1} x$$

Eulers method on the system $(**)$ is then

$$w_{n+1} = w_n + h \Lambda w_n$$

$$= (I + h\Lambda) w_n$$

Here $w_n \in \mathbb{C}^2$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

This difference equation has the solution

$$w_{n+1} = (I + h\Lambda)^{n+1} u_0$$

In the first component this is $w^1_{n+1} = (1+hi)^{n+1} u^1_0$

and in the second it is

$$w^2_{n+1} = (1-hi)^{n+1} u^2_0$$

Since $|1 \pm hi| > 1$

This means that the solution $w_{n+1} \to \infty$ as $n \to \infty$

If $y_{n+1} = y_n + Ay_n$

$$= (I + A) y_n$$

is the approximation of the original system $\dot{x} = Ax \quad x(0) = x$

We have $W_n = V^{-1} Y_n$

$\Rightarrow Y_n \to \infty$ as $n \to \infty$

as well.