MAT3110 FALL 2023 Oblig 1

Hand in deadline

Thursday, September 28, 2023, 14:30, uploaded to Canvas.

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). **The assignment must be submitted as a single PDF file.** Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all the necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. (Add the code to the single pdf.) You can use your programming language of choice.

There is only one attempt to pass the assignment and you must have a score of at least 60% to pass it.

Application for postponement

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail:`studieinfo@math.uio.no`) well before the deadline.

Both mandatory assignments in this course must be approved in the same semester before you are allowed to take the final examination.

Complete guidelines on compulsory assignments

For further details on the hand in of compulsory assignments, see:
`https://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html`

Problems

Problem 1.

In this problem, we will study properties of Newton's method when it is used to solve $f(x) = 0$ for $f(x) := \arctan(x)$. The equation has one solution: $\xi = 0$.

a) Let (x_k) be a sequence obtained by Newton's method for solving the above equation. Explain why $\lim_{k\to\infty} x_k = \xi$ if x_0 is sufficiently near ξ and show that the sequence converges with at least order $q = 3$.

   Hint:for the order of convergence it may help to use that $|f''(x)| \leq |x|$ combined with (1.23) from (SM).

b) Show that

$$g(x) := \begin{cases} f(x)/(xf'(x)) & \text{hvis} \quad x \neq 0 \\ 1 & \text{hvis} \quad x = 0 \end{cases}$$

   is a continuous and symmetric function (meaning that $g(x) = g(-x)$), and that Newton's method (for this particular problem) can be written kan skrives

$$x_{k+1} = x_k(1 - g(x_k)) \qquad k = 0, 1, \ldots \tag{1}$$

c) Show that there exists a unique point $x^* \in (0, \infty)$ such that $g(x^*) = 2$ and that

$$g(x) > 2 \quad \forall x \in (x^*, \infty).$$

   Hint: Show that $g(x)$ strictly increasing on $(0, \infty)$. You can use that $\arctan(x) \leq x$ for all $x > 0$.

d) Approximate the value of x^* numerically to 5 correct decimal digits using an iteration method.

e) Use the results in 1 b) and c) to show that the Newton sequence diverges if x_0 is far from ξ, and describe the largest possible non-empty interval (a, b) that contains ξ and satisfies that

$$\lim_{k\to\infty} x_k = \xi \quad \text{for all Newton sequences with} \quad x_0 \in (a, b).$$

   Hint: Use the representation (1) and show (alternatively use if it becomes difficult to show) that if $x_0 < x^*$, then $g(x_k) < g(x_0) < 2$ for all $k \geq 0$, and if $x_0 > x^*$, then $g(x_k) > g(x_0) > 2$ for all $k \geq 0$.

f) Compute the sequence $(x_k)_{k=0}^{6}$ with Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

   for to ulike startverdier: $x_0 = 1.3$ og $x_0 = 1.4$. (That is, do not use the representation (1) in your computer implementation, because that representation is challenging to implement correctly such that one avoids problems when $x_k = 0$.) Do your numerical results align with your theoretical result in 1 e)?

Problem 2

Compute the QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix},$$

and use the factorization to find the least squares solution of the following equation

$$Ax = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}_{=b}.$$

Problem 3

For a matrix $A \in \mathbb{R}^{n \times n}$ for $n \geq 2$, it costs $e^1 n! + \mathcal{O}((n-1)!)$ arithmetic operations to compute the determinant when using the formula

$$\det(A) = \sum_{\text{perm}} \text{sign}(\nu_1, \dots, \nu_n) a_{1\nu_1} a_{2\nu_2} \cdots a_{n\nu_n},$$

and additionally using co-factor matrices (see (SM) page 43). Even for small n-values, like $n = 100$, this computational cost becomes so high that the approach is useless as a computational algorithm. In this exercise, we will look at how matrix determinants in practice are computed far more efficiently in computer languages like Matlab.

Let $PA = LU$ be an LU factorization of A, where P is a permutation matrix, L is a unit lower triangular matrix and U is upper triangular. Explain how the matrices P, L and U can be used to construct a more efficient method for computing $\det(A)$ and compute the computational cost of your approach. You may use that the cost of the LU factorization of A is $2n^3/3 + \mathcal{O}(n^2)$ arithmetic operations, the cost of $\det(P)$ is $\mathcal{O}(n^2)$ arithmetic operations and that $\det(P) \neq 0$ (it always takes one of the values ± 1).