UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in MAT-INF 4130 — Numerical linear algebra

Day of examination: 12 December 2014

Examination hours: 0900 – 1300

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 9 part questions will be weighted equally.

Problem 1 Gauss-Seidel iteration

For a matrix $A \in \mathbb{R}^{n,n}$ with non-zero diagonal elements and vector $\boldsymbol{b} \in \mathbb{R}^n$, the Gauss-Seidel iteration for solving $A\boldsymbol{x} = \boldsymbol{b}$ is

$$L\boldsymbol{x}_{k+1} = \boldsymbol{b} - U\boldsymbol{x}_k, \qquad k = 0, 1, 2, \ldots,$$

with $\boldsymbol{x}_0 \in \mathbb{R}^n$ some initial guess, and $L, U \in \mathbb{R}^{n,n}$ the lower triangular and *strictly* upper triangular parts of A:

$$l_{ij} = \begin{cases} a_{ij}, & j \le i; \\ 0, & j > i, \end{cases} \qquad u_{ij} = \begin{cases} 0, & j \le i; \\ a_{ij}, & j > i. \end{cases}$$

1a

By expressing the iteration in the form

$$\boldsymbol{x}_{k+1} = G\boldsymbol{x}_k + \boldsymbol{c},$$

derive a sufficient conditon for convergence in terms of $\|G\|$ for some operator norm $\|\cdot\|$.

1b

If

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix},$$

how much does the error reduce in the ∞-norm after k iterations?

(Continued on page 2.)

1c

Write a matlab program for the Gauss-Seidel method applied to a matrix $A \in \mathbb{R}^{n,n}$ and right-hand side $\boldsymbol{b} \in \mathbb{R}^n$. Use the ∞-norm of the difference $\boldsymbol{x}_{k+1} - \boldsymbol{x}_k$ as the stopping criterion, as well as a maximum number of iterations. Try to write the routine without the storage of any vectors or matrices except the input initial guess $\boldsymbol{x}$ which can be overwritten with the approximate solution.

Problem 2 Least squares

2a

Suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_2 = \|\boldsymbol{y}\|_2$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Show that the Householder transformation

$$H = I - 2\frac{\boldsymbol{v}\boldsymbol{v}^T}{\boldsymbol{v}^T\boldsymbol{v}},$$

where $\boldsymbol{v} = \boldsymbol{x} - \boldsymbol{y}$, results in

$$H\boldsymbol{x} = \boldsymbol{y}.$$

2b

Let

$$A = \begin{bmatrix} 2 & 0 \\ 2 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}.$$

Compute a Householder transformation H such that the first column of HA has a zero in the last two positions.

2c

Use H to find a QR decomposition of A and a corresponding QR factorization, $A = Q_1 R_1$.

2d

Use Q_1 and R_1 to find the least squares soluton $\boldsymbol{x} \in \mathbb{R}^2$ to $A\boldsymbol{x} \approx \boldsymbol{b}$ (hint: you can use the fact that the normal equations are $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$).

Problem 3 Eigenvalues

Let A be the matrix

$$\begin{bmatrix} 4 & -4 \\ -1 & 4 \end{bmatrix}.$$

(Continued on page 3.)

3a

Find the eigenvalues and eigenvectors of A.

3b

Consider the iteration $\boldsymbol{z}_k = A\boldsymbol{z}_{k-1}$, for $k = 1, 2, 3, \ldots$, with $\boldsymbol{z}_0 \in \mathbb{R}^2$, and let $\boldsymbol{x}_k = \boldsymbol{z}_k / \|\boldsymbol{z}_k\|_2$. What does $\boldsymbol{x}_k$ converge to (mathematically) if (i) $\boldsymbol{z}_0 = [2, 0]^T$? (ii) $\boldsymbol{z}_0 = [2, 1]^T$?

Good luck!