EXERCISES MAT3110

November 22

Exercises from SM.

12.1
12.2 **Hint:** Note that $y(t) = 0$ is a solution.
12.4
12.6
12.19

Exercise 1. Explain why an explicit Runge–Kutta method cannot be A-stable.

Exercise 2.

a) Heun's method is the Runge–Kutta method with Butcher tableau

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

Write the associated one-step method on the form

$$y_{n+1} = y_n + h\Phi(t_n, y_n, y_{n+1}; h),$$

and determine a formula for the stepping rule $\Phi(t_n, y_n, y_{n+1}; h)$. Is the method explicit or implicit?

b) What is the method's region of absolute stability, and is the method A-stable?

c) Consider the IVP

$$\begin{aligned} y'(t) &= f(y) \qquad t \in [a, b] \\ y(a) &= y_0, \end{aligned}$$

where $f \in C^2(\mathbb{R})$ and suppose it has a unique solution $y \in C^3([a, b])$.

Show that for this particular problem, the above Runge–Kutta method is consistent and has order of accuracy $p = 2$. (It also is consistent and has order 2 in the more general sense: when applied to all IVP with sufficiently smooth solutions and $f(t, y)$. It takes a bit longer Taylor expansions in both arguments of $f(t, y)$ to show that.)

November 15

Exercises from SM.

- 10.1
- 10.2 **Hint:** What happens when you apply the Gauss quadrature to integrating $f = L_k \in \mathcal{P}_n$?
- 11.2 exercises (i) and (iii)
- 11.6
- 11.7

Exercise 1. Estimate the integral

$$I(f) = \int_{[0,1]^3} f(x) dx_1 dx_2 dx_3$$

where

$$f(x) = \cos(\pi x_2^2) \sin(\pi x_1 - x_3/2) + \exp(x_1 x_2)$$

using the Monte Carlo method on a computer. How many samples M are needed to ensure that the root mean square error is less than 10^{-4}?

Exercise 2. Let

$$I(f) = \int_{[0,1]} f(x) dx$$

with

$$f(x) = x^{-1/4}.$$

How many Monte Carlo samples are need to ensure that

$$\mathbb{P}(|I(f) - I_M(f)| \geq 10^{-5}) \leq 0.1 \quad ?$$

Hint: Compute or obtain an upper bound for $\mathrm{Var}[f(X)]$ and translate inequality (2) in "The Monte Carlo method in a nutshell" to the current setting.

Exercise 3. Compute the composite Gauss rule formula for $n = 1$ when $[a, b] = [-1, 1]$ and $w \equiv 1$. (That is, compute $G_{m,1}$ in the notation given below in (4).)

Exercise 4. (Long, not very course relevant exercise for those interested.) Here we seek to verify the error estimate for the composite Gauss rule that was handwavingly presented in the lecture for the special case with weight function $w \equiv 1$, namely:

$$\left| \int_a^b f(x) dx - G_{m,n} \right| \leq \frac{\max_{x \in [a,b]} |f^{(2n+2)}(x)|}{(2n+2)!} \left(\frac{h}{2} \right)^{2n+2} (b - a). \tag{1}$$

(The factor 2^{2n+2} that appears in the denominator above was written incorrectly as 2^{3n+3} in the lecture.) From the lecture, we recall that

$$G_n(a, b) = \sum_{k=0}^{n} W_k f(x_k)$$

denotes the Gauss rule over a given interval $[a, b]$ using the $n+1$ quadrature points $\{x_k\}_{k=0}^n$, and that

$$\left| \int_a^b f(x) dx - G_n(a, b) \right| \leq \frac{\max_{x \in [a,b]} |f^{(2n+2)}(x)|}{(2n+2)!} \int_a^b (\pi_{n+1}(x))^2 dx \tag{2}$$

where

$$\pi_{n+1}(x) = \prod_{k=0}^{n} (x - x_k).$$

Chopping up $[a,b]$ at the mesh points $x_i = a + ih$ with $h = (b-a)/m$, we obtain subintervals $[x_{i-1}, x_i]$ to produce the approximation

$$\int_a^b f(x)dx \approx \sum_{i=1}^m G_n(x_{i-1}, x_i) =: G_{m,n}.$$

This is the composite Gauss rule using m subintervals with $n+1$ quadrature points over each subinterval. Relating a point in the standard interval, $t \in [-1,1]$, uniquely to a point in a subinterval, $x \in [x_{i-1}, x_i]$, through the change of variables

$$t(x) = \frac{2x - (x_{i-1} + x_)}{h}, \quad x(t) = \frac{x_{i-1} + x_i + ht}{2} \tag{3}$$

(and connecting Legendre polynomials on $[-1,1]$ to a corresponding set of basis functions on $[x_{i-1}, x_i]$) we obtained that

$$G_{m,n} = \frac{h}{2} \sum_{i=1}^m \sum_{k=0}^n W_k f\left(\frac{x_{i-1} + x_i + t_k h}{2}\right) \tag{4}$$

with $\{t_k\}_{k=0}^n$ denoting the zeros of the Legendre polynomial of exact degree $n+1$ (i.e. on $[-1,1]$) and the weights W_k in the particular case of (4) being defined by

$$W_k = \int_{-1}^1 (L_k(t))^2 dt.$$

It was shown in the lecture that when $\{t_k\}_{k=0}^n$ are the Gauss quadrature points for $G_n(-1,1)$, then $\{x(t_k)\}_{k=0}^n$ according to the above change of variables are the Gauss quadradure points for $G_n(x_{-1}, x_i)$. Using this in (2), we obtain that

$$\left| \int_{x_{i-1}}^{x_i} f(x)dx - G_n(x_{i-1}, x_i) \right| \leq \frac{\max_{x \in [x_{i-1}, x_i]} |f^{(2n+2)}(x)|}{(2n+2)!} \int_{x_{i-1}}^{x_i} (\pi_{n+1,i}(x))^2 dx$$

with $\pi_{n+1,i}(x) := \prod_{k=0}^n (x - x(t_k))$. By the triangle inequality, this further gives that

$$\left| \int_a^b f(x)dx - G_{m,n} \right| \leq \sum_{i=1}^m \frac{\max_{x \in [x_{i-1}, x_i]} |f^{(2n+2)}(x)|}{(2n+2)!} \int_{x_{i-1}}^{x_i} (\pi_{n+1,i}(x))^2 dx \tag{5}$$

(where $x(t_k)$ maps to the interval $[x_{i-1}, x_i]$ of the index i that is "active" in the summation).

a) Use the change of variables (3) to show that

$$\int_a^b (\pi_{n+1,i}(x))^2 dx = \left(\frac{h}{2}\right)^{2n+3} \int_{-1}^1 (\pi_{n+1}(t))^2 dt$$

where $\pi_{n+1}(t) = \prod_{k=0}^n (t - t_k)$ for $t \in [-1,1]$.

b) Show that for any $n \geq 0$ and with $\{t_k\}_{k=0}^n$ as above denoting the roots of the Legendre polynomial of exact degree $n+1$, that it holds that $\max_{t \in [-1,1]} |\pi_{n+1}(t)| \leq 1$.

You may use the following fact without proof: The Legendre polynomial ϕ_n with exact degree n is an odd function when n is odd and an even function when n is even. (What does this imply about the all roots of ϕ_n except possibly a root at $t = 0$?)

c) Use parts a) and b) in combination with (5) to show that (1) holds.

NOVEMBER 8

Exercises from SM.

9.1
9.2
9.5
9.10

Exercise 5. Find the best approximation $p_1 \in \mathcal{P}_1$ in $L^2_w(0,1)$ with $w(x) = x$ to $f(x) = e^x$ and compute the mean square error $\|f - p_1\|_2^2$.

Exercise 6. Given a bounded interval (a, b), a weight function $w : (a, b) \to (0, \infty)$ (that by assumption is integrable) and the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)w(x)dx \qquad f, g \in L^2_w(a, b),$$

let $\phi_0, \phi_1, \dots$ be a system of polynomials that are orthogonal with respect to the given inner product with $\phi_0 \equiv 1$ and degree$(\phi_j) = j$. In this exercise we will show that for any $f \in L^2_w(a, b)$, there exists a sequence $(c_k) \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} \|p_n - f\|_2 = 0 \quad \text{for} \quad p_n(x) := \sum_{k=0}^{n} c_k \phi_k(x). \tag{6}$$

a) Show that for any $\tilde{f} \in C[a, b]$, it holds that

$$\|\tilde{f}\|_2 \le C \max_{x\in[a,b]} |\tilde{f}(x)| = C\|\tilde{f}\|_\infty$$

and find a suitable value for the constant C.

b) Show that for any $\tilde{f} \in C[a, b]$, there exists a sequence of polynomials $(\tilde{p}_n)$ with $\tilde{p}_n \in \mathcal{P}_n$ such that

$$\lim_{n\to\infty} \|\tilde{p}_n - \tilde{f}\|_2 = 0.$$

Hint: Use Weierstrass approximation theorem combined with part a).

c) Take as a fact that $C[a, b]$ is dense in $L^2_w(a, b)$, and use this to prove that for any $f \in L^2_w(a, b)$, there exists a sequence of polynomials $(\hat{p}_n)$ with $\hat{p}_n \in \mathcal{P}_n$ such that

$$\lim_{n\to\infty} \|\hat{p}_n - f\|_2 = 0.$$

Hint: Density implies there exists a sequence $(\tilde{f}_n) \subset C[a, b]$ such that $\|\tilde{f}_n - f\|_2 \to 0$ as $n \to \infty$. And for any fixed n, we know by b) that there exists a sequence of polynomials $(\tilde{p}_{n,m})_{m=1}^\infty$ with $\tilde{p}_{n,m} \in \mathcal{P}_m$ such that $\|\tilde{f}_n - \tilde{p}_{n,m}\|_2 \to 0$ as $m \to \infty$.

d) Show that the convergence (6) holds when

$$c_k = \frac{\langle f, \phi_k \rangle}{\langle \phi_k, \phi_k \rangle}.$$

Hint: For each $n \ge 0$, show first that (since p_n is the best approximation of degree n in 2-norm),

$$\|p_n - f\|_2 \le \|\hat{p}_n - f\|_2.$$

November 1

Exercises from SM.

8.2 **Hint:** If $p(x)$ is minmax to $f(x)$, what is a minmax to $\tilde{f}(x) = f(-x)$?
8.3
8.4
8.5
8.8 **Hint:** $f(x) = a_{n+1}x^{n+1} + \underbrace{\sum_{k=0}^{n} a_k x^k}_{\in \mathcal{P}_n}$

Exercise related to proof of Theorem 8.2 in SM. On the bottom of page 229, it is stated "the set $\mathcal{S}$ is evidently bounded and closed in $\mathbb{R}^{n+1}$", and on top of page 230, it is claimed there exits a minimizer c^* of $E(c) = \|f-(c_0+c_1x+\ldots+c_nx^n)\|_\infty$ over $\mathcal{S}$ such that

$$E(c^*) = \min_{c\in\mathcal{S}} E(c)$$

(In my proof in the lecture, the set $\mathcal{S}$ was instead called $\mathcal{A}$ and instead of c^* I let d denote the minimizer in $\mathcal{A}$, but the core open questions remain.)

We know that, for a continuous function, the preimage of a bounded set is again bounded. Since $E\colon \mathbb{R}^{n+1} \to [0,\infty)$ is a continuous function and $\mathcal{S} = E^{-1}([0, \|f\|_\infty + 1])$ is the preimage of the closed set $[0, \|f\|_\infty + 1]$, the set $\mathcal{S}$ is closed too.

The fact that $\mathcal{S}$ is also bounded and that the minimizer c^* of E over the set $\mathcal{S}$ is itself contained in $\mathcal{S}$ follows from further results in real analysis. However, since a course in real analysis is not a prerequisite for this course, we will prove these two results which are critical for the proof of Theorem 8.2 in this exercise.

a) Show that $\mathcal{S} = \{c \in \mathbb{R}^{n+1} \mid E(c) \leq \|f\|_\infty + 1\} = E^{-1}([0, \|f\|_\infty + 1])$, is bounded.

Hint: Show first that $\mathcal{S}$ is a convex set, meaning that if $c, d \in \mathcal{S}$ then $(\lambda c + (1-\lambda)d) \in \mathcal{S}$ for all $\lambda \in [0,1]$.

Suppose next that $\mathcal{S}$ is unbounded. Then the convexity of $\mathcal{S}$ and the knowledge that $0 \in \mathcal{S}$ imply that there exist a "direction" $\tilde{c} = (\tilde{c}_0, \ldots, \tilde{c}_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ such that $\mu\tilde{c} \in \mathcal{S}$ for all $\mu > 0$. Combine this with

$$\|f-q\|_\infty \geq \|q\|_\infty - \|f\|_\infty \qquad \forall q \in \mathcal{P}_n$$

to reach a contradiction.

b) Make use of the following fact from real analysis (not curriculum) – $\mathcal{S} \subset \mathbb{R}^{n+1}$ is compact implies that every sequence $(c^{(k)})$ in $\mathcal{S}$ has a subsequence $(c^{(k_j)})$ that converges to an element in $\mathcal{S}$ – to show that there exists a minimizer of E over $\mathcal{S}$ that belongs to $\mathcal{S}$, i.e.,

$$\exists c^* \in \mathcal{S} \quad \text{s.t.} \quad E(c^*) = \min_{c\in\mathcal{S}} E(c).$$

Hint: Consider a sequence $(c^{(k)}) \subset \mathcal{S}$ such that

$$\lim_{k\to\infty} E(c^{(k)}) = \inf_{c\mathcal{S}} E(c)$$

(which exists by definition). Exploit compactness of $\mathcal{S}$ and continuity of E to arrive at the conclusion.

October 25

Exercises from SM.

7.1

7.2 Hint: Theorem 7.1 and that any $p_{n+1} \in \mathcal{P}_{n+1}$ can be written as $p_{n+1} = c(x - (a+b)/2)^{n+1} + r_n$ for some $c \in \mathbb{R}$ and $r_n \in \mathcal{P}_n$.

7.3 Hint: The formula holding for all $f \in \mathcal{P}_1$ is equivalent to it holding for $f = 1$ and $f = x$.

7.4 Hint: extend hint from previous exercise.

7.7

7.13

October 18

1. Let $x_i = a + ih$ for $i = 0, \ldots, n$ with $h = (b-a)/n$ be equally spaced points over $[a, b]$ and for $n \geq 1$, consider the function

$$\pi_{n+1}^E(x) = \prod_{i=0}^{n}(x - x_i) \qquad x \in [a, b]$$

that we know from interpolation error estimates. For later reference in two weeks, we now use the superscript E in the name of π_{n+1}^E to stress that the $\{x_i\}$ are equally spaced points.

a) Show that

$$\max_{x \in [a,b]} |\pi_{n+1}^E(x)| \leq n! h^{n+1}.$$

Hint: Noting that $(x_i - a)/h = i$ and writing $t = (x-a)/h \in [0, n]$, we have that

$$\prod_{i=0}^{n} |x - x_i| = h^{n+1} \prod_{i=0}^{n} |t - i|$$

and with $\lfloor t \rfloor := \max\{y \in \mathbb{N} \mid y \leq t\}$, bound the two factors

$$\prod_{i=0}^{n} |t - i| = \left(\prod_{i=0}^{\lfloor t \rfloor} (t - i) \right) \left(\prod_{i=\lfloor t \rfloor + 1}^{n} (i - t) \right)$$

from above separately.

b) To sharpen this bound, suppose next that

$$\max_{x \in [a,b]} |\pi_{n+1}^E(x)| = \max_{x \in [a,x_1]} |\pi_{n+1}^E(x)|$$

(this is true but a bit messy to show; see for instance Isaacson and Keller, Analysis of Numerical methods, p. 267). Use this property to show that

$$\max_{x \in [a,b]} |\pi_{n+1}^E(x)| \geq h^{n+1} \frac{(2n)!}{2^{2n+1} n!}$$

and

$$\max_{x \in [a,b]} |\pi_{n+1}^E(x)| \leq h^{n+1} \frac{n!}{4}.$$

Hint: A lower bound is given by $|\pi_{n+1}(a + h/2)|$.
For the upper bound, first show that

$$\max_{x \in [a,x_1]} (x - x_0)(x_1 - x) \leq h^2/4$$

and use this together with another splitting of factors, similar as in a).

c) Use the following global Stirling's approximation bounds:

$$\sqrt{2\pi n} \left(\frac{n}{e} \right)^n \exp\left(\frac{1}{12n+1}\right) < n! < \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \exp\left(\frac{1}{12n}\right)$$

to show that the lower bound is indeed smaller or equal to the upper bound in b) for any $n \geq 2$ (and for $n = 1$ they coincide).

Exercises from SM.

6.1
6.2
6.3
6.5
6.10

October 11

1. Let $A \in \mathbb{R}^{n\times n}_{sym}$ with orthonormal eigenbasis $v_1, \ldots, v_n$ and corresponding eign-values $\lambda_1, \ldots, \lambda_n$.

a) Show that for the Rayleigh quotient

$$R(v) := \frac{v^T A v}{v^T v}$$

it holds for any $j = 1, \ldots, n$ that

$$R(v_j) = \lambda_j$$

b) Show that $\max_{i,j} |\lambda_i - \lambda_j| \leq 2\|A\|_2$.
Hint: For $i \neq j$, show that

$$\lambda_i - \lambda_j = R(v_i) - R(v_j) = \ldots = (v_i + v_j)^T A (v_i - v_j),$$

and use Cauchy–Schwarz inequality.

c) Show that for a perturbed eigenvector $\tilde{v} = v_j + \Delta v$ with

$$\Delta v = \sum_{\substack{i=1 \\ i\neq j}}^{n} \epsilon_i v_i$$

with $(\epsilon_i) \subset \mathbb{R}$, it holds that

$$|R(\tilde{v}) - \lambda_j| \leq 2\|A\|_2 \|\Delta v\|_2^2.$$

Hint: use orthogonality to expand numerator and denominator of $R(\tilde{v})$ and use part b) of exercise.

2. Show that for any subordinate matrix p-norm, $p \in [1, \infty]$ and diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n\times n}$, it holds that

$$\|D\|_p = \max_{i=1,\ldots,n} |d_i|.$$

Look through the proof of the Bauer–Fike theorem and convince yourself using the above property that inequality (5.11) in the lecture notes can be extended from the 2-norm to any p-norm as follows:

$$\min_{\lambda\in\sigma(A)} |\mu - \lambda| \leq \kappa_p(T)\|\Delta A\|_p.$$

Exercises from SM.

5.5 Hint: Write/separate both left-hand side and right-hand side of

$$(D + \varepsilon A)(e + \varepsilon u) = (\lambda + \varepsilon\mu)(e + \varepsilon u)$$

into sums of multiplied powers of ε. That is, terms multiplied with 1, terms multiplied with ε, and terms multiplied with ε^2. These constitute equations on different ε-scales. Solve them separately.

5.10

5.13 Hint: Bauer–Fike applied to symmetric matrix T and $T + \Delta T$, where $\Delta T \in \mathbb{R}^{n\times n}$ contains one non-zero element, namely $\Delta T_{n,n-1} \neq 0$.

October 4

1. Consider the nonlinear system of equations

$$f(x) = \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ \exp(x_1 x_2) + x_1 + x_2 - 1 \end{bmatrix}. \tag{7}$$

a) Using that $f_1(x_1, x_2) = 0 \implies x_2 = \pm\sqrt{1 - x_1^2}$ for $x_1 \in [-1, 1]$, show that plugging this into $f_2(x_1, x_2) = 0$ leads to the equations

$$\underbrace{x_1 \pm \sqrt{1 - x_1^2} + \exp(\pm x_1 \sqrt{1 - x_1^2}) - 1}_{=: g_\pm(x_1)} = 0 \qquad x_1 \in [-1, 1].$$

Moreover, show that $g_+(x) = 0$ has a unique solution $x_{1,+} \in (-1, 0)$ and $g_- = 0$ has a unique solution $x_{1,-} \in (0, 1)$. Conclude that the system of equations (7) has exactly two solutions.

Hint: For g_+, for instance, start with showing that $g'_+(x) > 0$ for all $x \in (-1, 0)$ and $g_+ > 0$ for all $x \geq 0$.

b) Compute the Jacobian of the function f, $J_f(x)$ and explain why J_f is nonsingular for x near either of the two solutions.

c) Find both solutions of the problem (7) using Newton's method with an (estimated) accuracy of 10^{-6} in $\infty-$norm (estimate the approximation error at k-th iteration by $\|x^{(k)} - x^{(k-1)}\|_\infty$). Estimate the order of convergence in your numerical experiments. Are your observations consistent with the order of convergence theory ensures?

Exercises from SM.

4.1
4.6
4.7
4.8 Hint: For second part of problem, compute $x^{(k+1)}$ explicitly when $x^{(k)} = (1 + \alpha, 1 - \alpha)$ to deduce linear convergence.

Problem 2 in Mat-Inf4130 Exam 2021.

September 27

1. For matrices in $\mathbb{R}^{n\times n}$ with $n \geq 2$ show that:

a) If U and $\tilde{U}$ are upper triangular, then the product $U\tilde{U}$ is also upper triangular.

(Hint: U^T and $\tilde{U}^T$ are lower triangular matrices.)

b) If U is upper triangular and invertible, then U^{-1} is upper triangular.

(Hint: You can use that for any invertible matrix A, it holds that $(A^T)^{-1} = (A^{-1})^T$.)

Exercises from SM.

3.1
3.3
3.4
3.7 Hint: Read first up on definition 3.4 in SM for banded matrices and impose the additional assumption that B nonsingular in this exercise (without this assumption exercise is not necessarily true).

September 20

1. Recall that a symmetric matrix $B \in \mathbb{R}^{n\times n}$ always has n real valued eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ that can be associated to an orthonormal set of eigenvectors $v_1, \ldots, v_n$, that thus spans $\mathbb{R}^n$. Use this property to prove that any $A \in \mathbb{R}^{n\times n}$,

$$\|A\|_2 = \sqrt{\sigma_n},$$

where σ_n denotes the largest eigenvalue of $A^T A$.

2. Let $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ be a matrix with $\text{rank}(A) = n$ and with QR factorization $A = QR$. Then for a vector $b \in \mathbb{R}^m$, the unique solution of the least squares problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2$$

can be obtained by solving the "QR problem"

$$Rx = Q^T b$$

and can also be obtained by the normal equations

$$A^T Ax = A^T b.$$

The condition numbers associated to the respective problems are $\kappa_2(R)$ and $\kappa_2(A^T A)$. Show that

$$\kappa_2(A^T A) = (\kappa_2(R))^2 \geq \kappa_2(R).$$

Note that this implies that the normal equations always is at least as ill-conditioned as the "QR problem".

Exercises from SM.

2.12
2.13 Hint: Find an equation for δx given that $Ax = b$ and use exercise 2.12.
2.14
2.15
Problem 1 in Mat-Inf4130 Exam 2021.

September 13

Let $n \geq 2$ be a natural number in the below exercises.

1. For $x \in \mathbb{R}^n$, explain why

$$\|x\|_p := \left(\sum_{k=1}^{n} |x_i|^p\right)^{1/p}$$

is not a norm for any $p \in (0, 1)$.

Hint: Show that not all three conditions for a norm is met.

2.

a) Using Hölder's inequality, show that for any $p > 1$ and any $x \in \mathbb{R}^n$,it holds that

$$\|x\|_1 \leq n^{(p-1)/p}\|x\|_p,$$

and that

$$\|x\|_p \leq n^{1/p}\|x\|_1,$$

(Hint: For the last inequality, first show directly, not using Hölder's inequality, that $\|x\|_p \leq n^{1/p}\|x\|_\infty$.)

b) Conclude from this that

$$n^{-1/p}\|x\|_p \leq \|x\|_1 \leq n^{(p-1)/p}\|x\|_p,$$

and any sequence $(x_k) \subset \mathbb{R}^n$, verify that the 1-norm and any p-norm for $p \in [1, \infty)$ are equivalent in the following sense

$$\lim_{k\to\infty} \|x_k - x^*\|_1 = 0 \iff \lim_{k\to\infty} \|x_k - x^*\|_p = 0.$$

Why does this imply the equivalence of any two norms $\|\cdot\|_p$ and $\|\cdot\|_q$ on $\mathbb{R}^n$ for $p, q \in [1, \infty)$?

c) Show that

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty,$$

and conclude from this and the previous exercise that all p-norms for $p \in [1, \infty]$ are equivalent.

3. For any vector norm $\|\cdot\|$ on $\mathbb{R}^n$, verify that the subordinate norm on $\mathbb{R}^{n\times n}$ given by

$$\|A\| := \max_{v\in\mathbb{R}^n_*} \frac{\|Av\|}{\|v\|}$$

is indeed a norm on $\mathbb{R}^{n\times n}$.

4. For any symmetric matrix $A \in \mathbb{R}^{n\times n}$, explain why $\|A\|_1 = \|A\|_\infty$. Find a 2×2 matrix such that

$$|A\|_1 \neq \|A\|_\infty.$$

Exercises from SM and exams.

2.7

2.8

2.9 Hint: Look at Theorem 2.9 in SM. What are the eigenvalues of the inverse of a matrix?

September 6

1. Let $n \geq 2$ and let $A \in \mathbb{R}^{n \times n}$ be such that $\det(A^{(i)}) \neq 0$ for all $1 \leq i < n$. Then we know from theory that there exists an LU factorization $A = LU$, where L is unit lower triangular and U is upper triangular. Such matrices L and U can be determined from the system of nonlinear equations

$$a_{ij} = \sum_{k=1}^{\min(i,j)} \ell_{ik} u_{kj} \qquad 1 \leq i, j \leq n.$$

This is a system of n^2 equations with n^2 unknowns. Use that $u_{ii} \neq 0$ for all $i \in \{1, \ldots, n-1\}$ (this is a consequence of $\det(A^{(n-1)} \neq 0)$ to show that the solution is unique.

Hint: Since $\ell_{11} = 1$, the first row of U is determined uniquely. And once u_{11} is determined, the first column of L is determined uniquely. Explain in a few sentences how by solving the equations sequentially, row/column by row/column, the solutions of L and U are unique.

2. Compute the LU factorization of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 9 \\ 3 & 10 & 18 \end{pmatrix}$$

From what you have obtained, it thus holds that $PA = LU$ with permutation matrix $P = I$. Explain why the matrix PA with

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

also has an LU factorization. Conclude from this that in general, there may not be a unique PLU triple (permutation, unit lower triangular and upper triangular) such that the factorization

$$PA = LU$$

holds.

Hint: How does P permute the order of rows in A?

Exercises from SM and exams.

2.1 Hint: If L is lower triangular and U is *unit* upper triangular, try to use the permutation matrix Q to write L and U in terms of $\widetilde{L}$ and $\widetilde{U}$, where $\widetilde{L}$ is *unit* lower triangular and $\widetilde{U}$ is upper triangular.

2.2 Hint: Observe that the product DU is upper triangular. Let A have a decomposition $A = L\widetilde{U}$. How do we find D and U such that $DU = \widetilde{U}$?

2.3

2.4

Problem 1d) in Mat-Inf4130 Exam 2013.