UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT 4350 — Numerical linear algebra

Day of examination: 7 December 2012

Examination hours: 0900 – 1300

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 9 part questions will be weighted equally.

Problem 1 Gauss-Seidel

Consider the matrix

$$\boldsymbol{A} := \begin{bmatrix} 4 & -\alpha \\ -\alpha & 1 \end{bmatrix}, \quad \alpha \in \mathbb{R}.$$

1a

For what values of α is $\boldsymbol{A}$ symmetric positive definite?

1b

For what values of α does Gauss Seidel's method converge?

Problem 2 Perturbation

Let $\| \ \|$ be a vector norm on $\mathbb{R}^n$ and for any $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ let

$$\|\boldsymbol{B}\| := \max_{\boldsymbol{x}\neq \boldsymbol{0}} \frac{\|\boldsymbol{Bx}\|}{\|\boldsymbol{x}\|}$$

be the associated operator norm of $\boldsymbol{B}$. Suppose $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is nonsingular.

2a

Show that for any $\boldsymbol{b}, \boldsymbol{e} \in \mathbb{R}^n$ with $\boldsymbol{b} \neq \boldsymbol{0}$

$$\frac{\|\boldsymbol{e}\|}{\|\boldsymbol{b}\|} \leq \|\boldsymbol{A}\|\|\boldsymbol{A}^{-1}\|\frac{\|\boldsymbol{y}-\boldsymbol{x}\|}{\|\boldsymbol{x}\|}, \tag{1}$$

(Continued on page 2.)

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are solutions of $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{Ay} = \boldsymbol{b} + \boldsymbol{e}$.
Hint: Use that $\boldsymbol{A}(\boldsymbol{y} - \boldsymbol{x}) = \boldsymbol{e}$ and $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

2b

Show that we have equality in (1) for some vectors $\boldsymbol{b}$ and $\boldsymbol{e}$.
Hint: There are vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ so that

$$\|\boldsymbol{A}^{-1}\| = \frac{\|\boldsymbol{A}^{-1}\boldsymbol{c}\|}{\|\boldsymbol{c}\|}, \quad \|\boldsymbol{A}\| = \frac{\|\boldsymbol{Ad}\|}{\|\boldsymbol{d}\|}.$$

You should not show this.

Problem 3 Eigenvalue bound

In this exercise we assume that $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ has eigenpairs $(\lambda_j, \boldsymbol{x}_j)$, $j = 1, \ldots, n$, where the eigenvector matrix $\boldsymbol{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ is nonsingular. We know that $\boldsymbol{A} = \boldsymbol{XDX}^{-1}$, where $\boldsymbol{D} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. We let $\|\boldsymbol{A}\|_2 := \max_{\boldsymbol{x}\neq\boldsymbol{0}} \|\boldsymbol{Ax}\|_2/\|\boldsymbol{x}\|_2$ be the spectral norm of $\boldsymbol{A}$.

We want to show the following theorem:

Theorem 1
To any $\mu \in \mathbb{R}$ with $\mu - \lambda_j \neq 0$ for $j = 1, \ldots, n.$ and $\boldsymbol{x} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_2 = 1$ we can find an eigenvalue λ of $\boldsymbol{A}$ such that

$$|\lambda - \mu| \leq K_2(\boldsymbol{X})\|\boldsymbol{r}\|_2,$$

where $\boldsymbol{r} := \boldsymbol{Ax} - \mu\boldsymbol{x}$ and $K_2(\boldsymbol{X}) := \|\boldsymbol{X}\|_2 \, \|\boldsymbol{X}^{-1}\|_2$.

3a

Show that $\|\boldsymbol{D}\|_2 = \rho(\boldsymbol{A}) := \max_i|\lambda_i|$.

3b

We define $\boldsymbol{D}_1 := \boldsymbol{D} - \mu\boldsymbol{I}$. Show that $\boldsymbol{D}_1$ is nonsingular and $\|\boldsymbol{D}_1^{-1}\|_2 = \frac{1}{\lambda-\mu}$, where $|\lambda - \mu| := \min_j|\lambda_j - \mu|$.

3c

Show that $\boldsymbol{XD}_1^{-1}\boldsymbol{X}^{-1}\boldsymbol{r} = \boldsymbol{x}$, where $\boldsymbol{r} := \boldsymbol{Ax} - \mu\boldsymbol{x}$.

3d

Show Theorem 1.

(Continued on page 3.)

Problem 4 Matlab program

Recall that a square matrix $\boldsymbol{A}$ is d-banded if $a_{ij} = 0$ for $|i - j| > d$. Write a Matlab `function x=backsolve(A,b,d)` that for a given nonsingular upper triangular d-banded matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$ computes a solution $\boldsymbol{x}$ to the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Good luck!