UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT 4350 — Numerical linear algebra

Day of examination: 7 December 2012

Examination hours: 0900 – 1300

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 9 part questions will be weighted equally.

Problem 1 Gauss-Seidel

Consider the matrix

$$\boldsymbol{A} := \begin{bmatrix} 4 & -\alpha \\ -\alpha & 1 \end{bmatrix}, \quad \alpha \in \mathbb{R}.$$

1a

For what values of α is $\boldsymbol{A}$ symmetric positive definite?

Answer: $\boldsymbol{A}$ is symmetric for any α. Since $a_{11} > 0$, the matrix $\boldsymbol{A}$ is positive definite if and only if $\det(\boldsymbol{A}) = 4 - \alpha^2 > 0$ or $-2 < \alpha < 2$.

1b

For what values of α does Gauss Seidel's method converge?

Answer: Applying GS to the system

$$\begin{bmatrix} 4 & -\alpha \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$$

we find $x_{k+1} = \frac{\alpha}{4} y_k + \frac{1}{4} b$ and $y_{k+1} = \alpha x_{k+1} + c = \alpha(\frac{\alpha}{4} y_k + \frac{1}{4} b) + c$. Thus

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \boldsymbol{G} \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \boldsymbol{c}, \quad \boldsymbol{G} = \begin{bmatrix} 0 & \alpha/4 \\ 0 & \alpha^2/4 \end{bmatrix}.$$

GS converges if and only if $\rho(\boldsymbol{G}) < 1$. Since $\boldsymbol{G}$ has eigenvalues 0 and $\alpha^2/4$ this happens if and only if $-2 < \alpha < 2$ i. e., if and only if $\boldsymbol{A}$ is positive definite.

(Continued on page 2.)

Problem 2 Perturbation

Let $\|\ \|$ be a vector norm on $\mathbb{R}^n$ and for any $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ let

$$\|\boldsymbol{B}\| := \max_{\boldsymbol{x}\neq \boldsymbol{0}} \frac{\|\boldsymbol{B}\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$$

be the associated operator norm of $\boldsymbol{B}$. Suppose $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is nonsingular.

2a

Show that for any $\boldsymbol{b}, \boldsymbol{e} \in \mathbb{R}^n$ with $\boldsymbol{b} \neq \boldsymbol{0}$

$$\frac{\|\boldsymbol{e}\|}{\|\boldsymbol{b}\|} \leq \|\boldsymbol{A}\|\|\boldsymbol{A}^{-1}\| \frac{\|\boldsymbol{y}-\boldsymbol{x}\|}{\|\boldsymbol{x}\|}, \tag{1}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are solutions of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} + \boldsymbol{e}$.
Hint: Use that $\boldsymbol{A}(\boldsymbol{y}-\boldsymbol{x}) = \boldsymbol{e}$ and $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$.

Answer: Subtracting $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ from $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} + \boldsymbol{e}$ we find $\boldsymbol{A}(\boldsymbol{y}-\boldsymbol{x}) = \boldsymbol{e}$. Taking norms

$$\|\boldsymbol{e}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{y}-\boldsymbol{x}\|, \quad \|\boldsymbol{x}\| \leq \|\boldsymbol{A}^{-1}\|\|\boldsymbol{b}\|.$$

But then $\frac{1}{\|\boldsymbol{b}\|} \leq \frac{\|\boldsymbol{A}^{-1}\|}{\|\boldsymbol{x}\|}$,

$$\frac{\|\boldsymbol{e}\|}{\|\boldsymbol{b}\|} \leq \|\boldsymbol{A}\|\|\boldsymbol{y}-\boldsymbol{x}\|\frac{\|\boldsymbol{A}^{-1}\|}{\|\boldsymbol{x}\|}.$$

and (1) follows.

2b

Show that we have equality in (1) for some vectors $\boldsymbol{b}$ and $\boldsymbol{e}$.
Hint: There are vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ so that

$$\|\boldsymbol{A}^{-1}\| = \frac{\|\boldsymbol{A}^{-1}\boldsymbol{c}\|}{\|\boldsymbol{c}\|}, \quad \|\boldsymbol{A}\| = \frac{\|\boldsymbol{A}\boldsymbol{d}\|}{\|\boldsymbol{d}\|}.$$

You should not show this.

Answer: Define $\boldsymbol{b} := \boldsymbol{c}$ and $\boldsymbol{e} := \boldsymbol{A}\boldsymbol{d}$. Then $\|\boldsymbol{e}\|/\|\boldsymbol{b}\| = \|\boldsymbol{A}\boldsymbol{d}\|/\|\boldsymbol{c}\|$. Now

$$\frac{1}{\|\boldsymbol{c}\|} = \frac{\|\boldsymbol{A}^{-1}\|}{\|\boldsymbol{A}^{-1}\boldsymbol{c}\|} = \frac{\|\boldsymbol{A}^{-1}\|}{\|\boldsymbol{x}\|}, \quad \|\boldsymbol{A}\boldsymbol{d}\| = \|\boldsymbol{A}\|\|\boldsymbol{d}\| = \|\boldsymbol{A}\|\|\boldsymbol{A}^{-1}\boldsymbol{e}\| = \|\boldsymbol{A}\|\|\boldsymbol{y}-\boldsymbol{x}\|.$$

But then

$$\frac{\|\boldsymbol{e}\|}{\|\boldsymbol{b}\|} = \frac{\|\boldsymbol{A}\boldsymbol{d}\|}{\|\boldsymbol{c}\|} = \frac{\|\boldsymbol{A}^{-1}\|}{\|\boldsymbol{x}\|}\|\boldsymbol{A}\|\|\boldsymbol{y}-\boldsymbol{x}\|$$

and (1) holds with equality.

(Continued on page 3.)

Problem 3 Eigenvalue bound

In this exercise we assume that $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ has eigenpairs $(\lambda_j, \boldsymbol{x}_j)$, $j = 1,\ldots,n$, where the eigenvector matrix $\boldsymbol{X} = [\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n]$ is nonsingular. We know that $\boldsymbol{A} = \boldsymbol{X}\boldsymbol{D}\boldsymbol{X}^{-1}$, where $\boldsymbol{D} = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$. We let $\|\boldsymbol{A}\|_2 := \max_{\boldsymbol{x}\neq\boldsymbol{0}}\|\boldsymbol{A}\boldsymbol{x}\|_2/\|\boldsymbol{x}\|_2$ be the spectral norm of $\boldsymbol{A}$.

We want to show the following theorem:

Theorem 1
To any $\mu \in \mathbb{R}$ with $\mu - \lambda_j \neq 0$ for $j = 1,\ldots,n$. and $\boldsymbol{x} \in \mathbb{R}^n$ with $\|\boldsymbol{x}\|_2 = 1$ we can find an eigenvalue λ of $\boldsymbol{A}$ such that

$$|\lambda - \mu| \leq K_2(\boldsymbol{X})\|\boldsymbol{r}\|_2,$$

where $\boldsymbol{r} := \boldsymbol{A}\boldsymbol{x} - \mu\boldsymbol{x}$ and $K_2(\boldsymbol{X}) := \|\boldsymbol{X}\|_2\,\|\boldsymbol{X}^{-1}\|_2$.

3a

Show that $\|\boldsymbol{D}\|_2 = \rho(\boldsymbol{A}) := \max_i|\lambda_i|$.

Answer: Since $\|\boldsymbol{D}\|_2$ equals the square root of the largest eigenvalue of $\boldsymbol{D}^T\boldsymbol{D}$ we have

$$\|\boldsymbol{D}\|_2 = \sqrt{\rho(\boldsymbol{D}^T\boldsymbol{D})} = \sqrt{\rho(\boldsymbol{D}^2)} = \sqrt{\rho(\boldsymbol{D})^2} = \rho(\boldsymbol{D}) = \rho(\boldsymbol{A}).$$

3b

We define $\boldsymbol{D}_1 := \boldsymbol{D} - \mu\boldsymbol{I}$. Show that $\boldsymbol{D}_1$ is nonsingular and $\|\boldsymbol{D}_1^{-1}\|_2 = \frac{1}{\lambda-\mu}$, where $|\lambda - \mu| := \min_j|\lambda_j - \mu|$.

Answer: $\boldsymbol{D}_1$ is nonsingular since it is a diagonal matrix with nonzero diagonal elements $\lambda_j - \mu$ for $j = 1,\ldots,n$. We have $\boldsymbol{D}_1^{-1} = \mathrm{diag}\big((\lambda_1 - \mu)^{-1},\ldots,(\lambda_n-\mu)^{-1}\big)$. The result follows from problem **3a** with $\boldsymbol{D} = \boldsymbol{D}_1$.

3c

Show that $\boldsymbol{X}\boldsymbol{D}_1^{-1}\boldsymbol{X}^{-1}\boldsymbol{r} = \boldsymbol{x}$, where $\boldsymbol{r} := \boldsymbol{A}\boldsymbol{x} - \mu\boldsymbol{x}$.

Answer:

$$\boldsymbol{X}\boldsymbol{D}_1^{-1}\boldsymbol{X}^{-1}\boldsymbol{r} = \big(\boldsymbol{X}(\boldsymbol{D} - \mu\boldsymbol{I})\boldsymbol{X}^{-1}\big)^{-1}\boldsymbol{r} = (\boldsymbol{A} - \mu\boldsymbol{I})^{-1}(\boldsymbol{A} - \mu\boldsymbol{I})\boldsymbol{x} = \boldsymbol{x}.$$

3d

Show Theorem 1.

Answer: Let $|\lambda - \mu| = \min_j|\lambda_j - \mu|$.

$$1 = \|\boldsymbol{x}\|_2 = \|\boldsymbol{X}\boldsymbol{D}_1^{-1}\boldsymbol{X}^{-1}\boldsymbol{r}\|_2 \leq \|\boldsymbol{D}_1^{-1}\|_2 K_2(\boldsymbol{X})\|\boldsymbol{r}\|_2 = \frac{K_2(\boldsymbol{X})\|\boldsymbol{r}\|_2}{\min_j|\lambda_j - \mu|} = \frac{K_2(\boldsymbol{X})\|\boldsymbol{r}\|_2}{|\lambda - \mu|}.$$

But then the theorem follows.

(Continued on page 4.)

Problem 4 Matlab program

Recall that a square matrix $\boldsymbol{A}$ is d-banded if $a_{ij} = 0$ for $|i - j| > d$. Write a Matlab `function x=backsolve(A,b,d)` that for a given nonsingular upper triangular d-banded matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}^n$ computes a solution $\boldsymbol{x}$ to the system $\boldsymbol{Ax} = \boldsymbol{b}$.

Answer:

```
function x=backsolve(A,b,d)
n=length(b); x=b;
x(n)=b(n)/A(n,n);
for k=n-1:-1:1
    uk=min(n,k+d);
    x(k)=(b(k)-A(k,k+1:uk)*x(k+1:uk))/A(k,k);
end
```

Good luck!