UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT-IN3110 — Introduction to numerical analysis

Day of examination: 5 December 2017

Examination hours: 0900 – 1300

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 9 part questions will be weighted equally.

Problem 1 Conditioning

Let $\| \; \|$ be a vector norm on $\mathbb{R}^n$ and for any $B \in \mathbb{R}^{n\times n}$ let

$$\|B\| := \max_{\mathbf{x}\neq 0} \frac{\|B\mathbf{x}\|}{\|\mathbf{x}\|}$$

be the associated operator norm of B. Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular.

Show that for any $\mathbf{b}, \mathbf{e} \in \mathbb{R}^n$ with $\mathbf{b} \neq 0$

$$\frac{\|\mathbf{e}\|}{\|\mathbf{b}\|} \leq \|A\|\|A^{-1}\| \frac{\|\mathbf{y}-\mathbf{x}\|}{\|\mathbf{x}\|},$$

where $\mathbf{x}$ and $\mathbf{y}$ are solutions of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b} + \mathbf{e}$.
Hint: Observe that $A(\mathbf{y} - \mathbf{x}) = \mathbf{e}$ and $\mathbf{x} = A^{-1}\mathbf{b}$.

Problem 2 LU

Find the LU factorization of the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Problem 3 Matlab program

Recall that a square matrix A is d-banded if $a_{ij} = 0$ for $|i - j| > d$. Write a Matlab `function x=backsolve(A,b,d)` that for a given nonsingular upper triangular d-banded matrix $A \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$ computes a solution $\mathbf{x}$ to the linear system $A\mathbf{x} = \mathbf{b}$.

(Continued on page 2.)

Problem 4 Polynomial interpolation

The divided difference $f[x_0, x_1, \ldots, x_n]$ of a function f at the distinct points $x_0, x_1, \ldots, x_n$ is the leading coefficient of the polynomial p of degree at most n that interpolates f at $x_0, x_1, \ldots, x_n$. Using the Lagrange form of p, or otherwise, find $f[x_0, x_1, \ldots, x_n]$ as a linear combination of $f(x_0), f(x_1), \ldots, f(x_n)$.

Problem 5 A non-linear equation

5a

We want to solve the equation

$$f(x) = x^2 - A = 0, \tag{1}$$

for x where $A > 0$ is a given constant. If $\{x_k\}$ is the sequence generated by Newton's method, show that

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{A}{x_k}\right), \qquad k = 0, 1, 2, \ldots.$$

5b

If $x_* > 0$ is the root of f in (1), and the k-th error is $e_k = x_k - x_*$, show that

$$e_{k+1} = e_k^2\left(\frac{1}{2x_k}\right), \qquad k = 0, 1, 2, \ldots.$$

5c

Suppose $1/4 \le A \le 1$ and that the initial guess is $x_0 = 1$. Show (a) that

$$x_* \le x_{k+1} \le x_k, \qquad k = 0, 1, 2, \ldots$$

and (b)

$$e_k \le e_0^{(2^k)}, \qquad k = 0, 1, 2, \ldots$$

Problem 6 Steepest descent

Suppose

$$F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}$$

from some positive definite matrix $A \in \mathbb{R}^{n\times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$, Suppose we want to find the unique minimum $\mathbf{x}_* \in \mathbb{R}^n$ of F using a descent method. If $\mathbf{x}$ is the current approximation to $\mathbf{x}_*$, the next approximation has the form

$$\mathbf{x}' = \mathbf{x} + \omega\mathbf{d},$$

(Continued on page 3.)

where $\mathbf{d}$ is the search direction.

If

$$\mathbf{g} = \nabla F(\mathbf{x}) = A\mathbf{x} - \mathbf{b}$$

denotes the gradient of F at $\mathbf{x}$, what is the descent condition on $\mathbf{d}$? Assuming the descent condition holds, find $\mathbf{x}'$ by minimizing F in the direction $\mathbf{d}$.

Problem 7 Fourier series

What is the complex Fourier series $f_N(t)$ of order N, with respect to the period $T > 0$, of a suitable function $f(t)$? Find $f_N(t)$ for

$$f(t) = \cos(4\pi t/T) + 3\sin(10\pi t/T + \pi/2),$$

when (a) $N = 2$ and (b) $N = 8$. Hint: you don't need integration.

Good luck!