UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT-IN3110 — Introduction to numerical analysis

Day of examination: 5 December 2017

Examination hours: 0900 – 1300

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 9 part questions will be weighted equally.

Problem 1 Conditioning

Let $\| \ \|$ be a vector norm on $\mathbb{R}^n$ and for any $B \in \mathbb{R}^{n\times n}$ let

$$\|B\| := \max_{\mathbf{x}\neq 0} \frac{\|B\mathbf{x}\|}{\|\mathbf{x}\|}$$

be the associated operator norm of B. Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular.

Show that for any $\mathbf{b}, \mathbf{e} \in \mathbb{R}^n$ with $\mathbf{b} \neq 0$

$$\frac{\|\mathbf{e}\|}{\|\mathbf{b}\|} \leq \|A\|\|A^{-1}\| \frac{\|\mathbf{y}-\mathbf{x}\|}{\|\mathbf{x}\|}, \tag{1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are solutions of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b} + \mathbf{e}$.
Hint: Use that $A(\mathbf{y} - \mathbf{x}) = \mathbf{e}$ and $\mathbf{x} = A^{-1}\mathbf{b}$.

Answer: Subtracting $A\mathbf{x} = \mathbf{b}$ from $A\mathbf{y} = \mathbf{b} + \mathbf{e}$ we find $A(\mathbf{y} - \mathbf{x}) = \mathbf{e}$. Taking norms

$$\|\mathbf{e}\| \leq \|A\|\|\mathbf{y}-\mathbf{x}\|, \quad \|\mathbf{x}\| \leq \|A^{-1}\|\|\mathbf{b}\|.$$

But then $\frac{1}{\|\mathbf{b}\|} \leq \frac{\|A^{-1}\|}{\|\mathbf{x}\|}$,

$$\frac{\|\mathbf{e}\|}{\|\mathbf{b}\|} \leq \|A\|\|\mathbf{y}-\mathbf{x}\| \frac{\|A^{-1}\|}{\|\mathbf{x}\|}.$$

and (1) follows.

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Problem 2 LU

Find the LU factorization of the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Answer Let

$$u_1^T = \begin{bmatrix} 3 & 4 \end{bmatrix}, \qquad l_1 = \frac{1}{3}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5/3 \end{bmatrix}.$$

Then

$$l_1 u_1^T = \begin{bmatrix} 3 & 4 \\ 5 & 20/3 \end{bmatrix}.$$

Then we let

$$A_1 = A - l_1 u_1^T = \begin{bmatrix} 0 & 0 \\ 0 & -2/3 \end{bmatrix}.$$

So now we set

$$u_2^T = \begin{bmatrix} 0 & -2/3 \end{bmatrix}, \qquad l_2 = -\frac{3}{2}\begin{bmatrix} 0 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then $A = LU$ where

$$L = \begin{bmatrix} l_1 & l_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5/3 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & -2/3 \end{bmatrix}.$$

Problem 3 Matlab program

Recall that a square matrix A is d-banded if $a_{ij} = 0$ for $|i - j| > d$. Write a Matlab `function x=backsolve(A,b,d)` that for a given nonsingular upper triangular d-banded matrix $A \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$ computes a solution $\mathbf{x}$ to the linear system $A\mathbf{x} = \mathbf{b}$.

Answer:

```
function x=backsolve(A,b,d)
n=length(b); x=b;
x(n)=b(n)/A(n,n);
for k=n-1:-1:1
    uk=min(n,k+d);
    x(k)=(b(k)-A(k,k+1:uk)*x(k+1:uk))/A(k,k);
end
```

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Problem 4 Polynomial interpolation

The divided difference $f[x_0, x_1, \ldots, x_n]$ of a function f at the distinct points $x_0, x_1, \ldots, x_n$ is the leading coefficient of the polynomial p of degree at most n that interpolates f at $x_0, x_1, \ldots, x_n$. Using the Lagrange form of p, or otherwise, find $f[x_0, x_1, \ldots, x_n]$ as a linear combination of $f(x_0), f(x_1), \ldots, f(x_n)$.

Answer: The Lagrange form of p is

$$p(x) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} f(x_i).$$

Therefore, writing

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots,$$

we see that the leading coefficient of p is

$$c_n = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{1}{x_i - x_j} f(x_i),$$

which is therefore the desired formula.

Problem 5 A non-linear equation

5a

We want to solve the equation

$$f(x) = x^2 - A = 0, \tag{2}$$

for x where $A > 0$ is a given constant. If $\{x_k\}$ is the sequence generated by Newton's method, show that

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{A}{x_k}\right), \qquad k = 0, 1, 2, \ldots. \tag{3}$$

Answer: Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, 2, \ldots.$$

Since $f(x_k) = x_k^2 - A$ and $f'(x_k) = 2x_k$, we have

$$\frac{f(x_k)}{f'(x_k)} = \frac{x_k}{2} - \frac{A}{2x_k},$$

from which the formula follows.

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5b

If $x_* > 0$ is the root of f in (2), and the k-th error is $e_k = x_k - x_*$, show that

$$e_{k+1} = e_k^2 \left(\frac{1}{2x_k} \right), \qquad k = 0, 1, 2, \ldots. \tag{4}$$

Answer: Since $A = x_*^2$, we find

$$\begin{aligned} e_{k+1} &= \frac{1}{2}\left(x_k + \frac{x_*^2}{x_k} \right) - x_* \\ &= \frac{1}{2x_k}\left(x_k^2 + x_*^2 - 2x_k x_* \right) \\ &= \frac{1}{2x_k}(x_k - x_*)^2 = \frac{1}{2x_k} e_k^2. \end{aligned}$$

5c

Suppose $1/4 \leq A \leq 1$ and that the initial guess is $x_0 = 1$. Show (a) that

$$x_* \leq x_{k+1} \leq x_k, \qquad k = 0, 1, 2, \ldots$$

and (b)

$$e_k \leq e_0^{(2^k)}, \qquad k = 0, 1, 2, \ldots$$

Answer: (a) If $1/4 \leq A \leq 1$ then $1/2 \leq x_* \leq 1$. By (4), $e_1 \geq 0$. This means that $x_1 \geq x_* > 0$. Then by (4) again, $e_2 \geq 0$. Continuing in this way we see that $e_k \geq 0$ for all k. From (3), we find that

$$x_{k+1} - x_k = \frac{1}{2x_k}(x_*^2 - x_k^2) \leq 0,$$

and so $x_{k+1} \leq x_k$.

(b) Since $x_k \geq x_* \geq 1/2$, by (4), $e_{k+1} \leq e_k^2$, and iterating this gives $e_k \leq e_0^{(2^k)}$.

Problem 6 Steepest descent

Suppose

$$F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A \mathbf{x} - \mathbf{x}^T \mathbf{b}$$

from some positive definite matrix $A \in \mathbb{R}^{n \times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$, Suppose we want to find the unique minimum $\mathbf{x}_* \in \mathbb{R}^n$ of F using a descent method. If $\mathbf{x}$ is the current approximation to $\mathbf{x}_*$, the next approximation has the form

$$\mathbf{x}' = \mathbf{x} + \omega \mathbf{d},$$

where $\mathbf{d}$ is the search direction.

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If

$$\mathbf{g} = \nabla F(\mathbf{x}) = A\mathbf{x} - \mathbf{b}$$

denotes the gradient of F at $\mathbf{x}$, what is the descent condition on $\mathbf{d}$? Assuming the descent condition holds, find $\mathbf{x}'$ by minimizing F in the direction $\mathbf{d}$.

Answer: The descent condition is that

$$\left[\frac{d}{d\omega}F(\mathbf{x}+\omega\mathbf{d})\right]_{\omega=0} = \mathbf{d}^T\mathbf{g} < 0.$$

We minimize the quadratic polynomial $F(\mathbf{x}+\omega\mathbf{d})$ with respect to ω. By the definition of F,

$$\begin{aligned} F(\mathbf{x}+\omega\mathbf{d}) &= \frac{1}{2}(\mathbf{x}+\omega\mathbf{d})^T A(\mathbf{x}+\omega\mathbf{d}) - (\mathbf{x}+\omega\mathbf{d})^T\mathbf{b} \\ &= \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \omega\mathbf{d}^T A\mathbf{x} + \frac{1}{2}\omega^2\mathbf{d}^T A\mathbf{d} - \mathbf{x}^T\mathbf{b} - \omega\mathbf{d}^T\mathbf{b} \\ &= F(\mathbf{x}) + \omega\mathbf{d}^T\mathbf{g} + \frac{1}{2}\omega^2\mathbf{d}^T A\mathbf{d}. \end{aligned}$$

The minimum of the quadratic is attained when

$$\frac{d}{d\omega}F(\mathbf{x}+\omega\mathbf{d}) = 0,$$

i.e., when

$$\mathbf{d}^T\mathbf{g} + \omega\mathbf{d}^T A\mathbf{d} = 0,$$

which implies that

$$\omega = -\frac{\mathbf{d}^T\mathbf{g}}{\mathbf{d}^T A\mathbf{d}},$$

which is positive if the descent condition holds. Thus

$$\mathbf{x}' = \mathbf{x} - \frac{\mathbf{d}^T\mathbf{g}}{\mathbf{d}^T A\mathbf{d}}\mathbf{d}.$$

Problem 7 Fourier series

What is the complex Fourier series $f_N(t)$ of order N, with respect to the period $T > 0$, of a suitable function $f(t)$? Find $f_N(t)$ for

$$f(t) = \cos(4\pi t/T) + 3\sin(10\pi t/T + \pi/2),$$

when (a) $N = 2$ and (b) $N = 8$. Hint: you don't need integration.

Answer: The complex Fourier series of f is

$$f_N(t) = \sum_{n=-N}^{N} y_n e^{2\pi int/T},$$

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where the coefficients y_n are such that f_N is the best L_2 approximation to f in the interval $[0, T]$, i.e.,

$$y_n = \frac{1}{T}\int_0^T f(t)e^{-2\pi int/T}\,dt.$$

For the given f we can find the coefficients y_n directly;

$$\begin{aligned} f(t) &= \cos(4\pi t/T) + 3\cos(10\pi t/T) \\ &= \frac{1}{2}(e^{2\pi i2t/T} + e^{-2\pi i2t/T}) + \frac{3}{2}(e^{5\pi i2t/T} + e^{-5\pi i2t/T}), \end{aligned}$$

and so $y_2 = y_{-2} = 1/2$ and and $y_5 = y_{-5} = 3/2$ and all other coefficients are zero. So (a):

$$f_2(t) = \frac{1}{2}e^{2\pi i2t/T} + \frac{1}{2}e^{-2\pi i2t/T},$$

and

$$f_8(t) = \frac{1}{2}e^{2\pi i2t/T} + \frac{1}{2}e^{-2\pi i2t/T} + \frac{3}{2}e^{5\pi i2t/T} + \frac{3}{2}e^{-5\pi i2t/T}.$$

Good luck!