UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3110 — Introduction to numerical analysis

Day of examination: 14 December 2018

Examination hours: 0900 – 1300

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 8 part questions will be weighted equally.

Problem 1 Matlab function

Write a Matlab (or Python) function `[L,U] = mylu(A)` that computes the LU factorization of an $n \times n$ matrix A, assuming that no pivoting is required.

Problem 2 Cholesky

Use the Cholesky algorithm to determine whether the matrix

$$A = \begin{bmatrix} 2 & 6 & -4 \\ 6 & 17 & 7 \\ -4 & 7 & 10 \end{bmatrix}$$

is positive-definite.

Problem 3 Matrix norm and SVD

3a

Recall that the 2-norm of a vector $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ is

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

If U is an orthogonal $n \times n$ matrix, what is the 2-norm of $U\mathbf{x}$?

(Continued on page 2.)

3b

What is the definition of the 2-norm of a real $n \times n$ matrix A?

3c

Using the singular value decomposition of A, show that the 2-norm of A equals σ_1, the largest singular value of A.

Problem 4 Iterative scheme

Consider the linear system $A\mathbf{x} = \mathbf{b}$ where A is a non-singular $n \times n$ matrix and $\mathbf{b}$ is a vector in $\mathbb{R}^n$. Suppose we try to find the solution $\mathbf{x}$ using the iterative scheme

$$(A - B)\mathbf{x}^{(k+1)} = -B\mathbf{x}^{(k)} + \mathbf{b}, \qquad k = 0, 1, 2,$$

with $\mathbf{x}^{(0)}$ some initial guess. Assuming $A - B$ is non-singular, what condition on A, B, and $\mathbf{b}$ is both necessary and sufficient for the sequence $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ to converge to $\mathbf{x}$? Explain your answer.

Problem 5 Polynomial interpolation

The polynomial of degree $\leq n$ that interpolates a function $f : [-1, 1] \to \mathbb{R}$ at distinct points $x_0, x_1, \ldots, x_n \in [-1, 1]$ can be expressed as

$$p_n(x) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} f(x_i).$$

For approximating f, what is a good choice of the points x_i when n is large?

Problem 6 Non-linear least squares

Suppose we want to minimize a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^{m} r_i(\mathbf{x})^2, \quad \mathbf{x} \in \mathbb{R}^n,$$

where $r_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$, are the so-called residuals. What are the Newton and Gauss-Newton methods for this problem, and what are their advantages and disadvantages?

Good luck!