UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3110 — Introduction to numerical analysis

Day of examination: 18 December 2019

Examination hours: 0900 – 1300

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 Matlab function

Write a Matlab (or Python) function `[L,D] = myCholesky(A)` that computes a Cholesky factorization $A = LDL^T$ of an $n \times n$ symmetric, positive definite matrix A.

Problem 2 QR

Consider the least squares solution to $A\mathbf{x} \approx \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 4 \\ 0 & 3 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix}.$$

2a

Find a reduced QR factorization of A, i.e., $A = Q_1 R_1$ for $Q_1 \in \mathbb{R}^{3,2}$ and $R_1 \in \mathbb{R}^{2,2}$, where Q has orthonormal columns and R_1 is upper triangular.

2b

Using Q_1 and R_1 find the solution $\mathbf{x}$.

(Continued on page 2.)

Problem 3 SVD

Consider the matrix

$$A = \begin{bmatrix} 1 & 8/3 \\ 0 & 1 \end{bmatrix}.$$

3a

Given that one eigenpair $(\lambda_1, \mathbf{v}_1)$ of $A^T A$ is

$$\lambda_1 = 9, \quad \mathbf{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

find the other eigenpair $(\lambda_2, \mathbf{v}_2)$, with $\mathbf{v}_2$ normalized.

3b

By letting

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \text{and} \quad U = AVD^{-1/2}$$

(or otherwise), find an SVD of A.

Problem 4 Interpolation

Let f be a function with four continuous derivatives in the interval $I = [-2h, 2h]$, for some $h > 0$. Let $p(x)$ be the polynomial of degree ≤ 3 such that $p(jh) = f(jh)$ for $j = -2, -1, 1, 2$. Show that for $x \in [-h, h]$,

$$|f(x) - p(x)| \leq \frac{1}{6} h^4 M_4, \quad \text{where} \quad M_4 = \max_{x \in I} |f^{(4)}(x)|.$$

Problem 5 Bernstein basis

A Bézier curve of degree n in $\mathbb{R}^2$ is a parametric curve $\mathbf{p}(t) = \sum_{i=0}^{n} B_{i,n}(t)\mathbf{c}_i$ with control points $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{R}^2$ and Bernstein basis functions

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i},$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$. The first derivative $\mathbf{p}'(t)$ can be expressed as a Bézier curve of degree $n-1$. What are its control points? Explain your answer.

Problem 6 Multivariate Newton's method

6a

Write down Newton's method for minimizing a function $f(\mathbf{x})$, where $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$.

(Continued on page 3.)

6b

Suppose $f(\mathbf{x}) = x^4 + 2x^2y^2 + y^4$. If $\mathbf{x} = (a, a)$ for some $a \in \mathbb{R}$, show that the gradient and Hessian of f at $\mathbf{x}$ are

$$\nabla f(\mathbf{x}) = 8a^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad Hf(\mathbf{x}) = 8a^2 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

6c

If the start vector is $\mathbf{x}^{(0)} = (a, a)$ for some $a \neq 0$, find the next iterate, $\mathbf{x}^{(1)}$, and deduce the rate of convergence to the global minimizer $(0, 0)$ in this case.

Good luck!