UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3110 — Introduction to numerical analysis

Day of examination: 18 December 2019

Examination hours: 0900 – 1300

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 Matlab function

Write a Matlab (or Python) function `[L,D] = myCholesky(A)` that computes a Cholesky factorization $A = LDL^T$ of an $n \times n$ symmetric, positive definite matrix A.

Answer:

```
function [L,D] = myCholesky(A)

n = size(A,1);
L=zeros(n,n);
D=zeros(n,n);
B = A;

for k=1:n
  L(:,k) = B(:,k) / B(k,k);
  D(k,k) = B(k,k);
  B = B - D(k,k) * L(:,k) * (L(:,k))';
end
```

Problem 2 QR

Consider the least squares solution to $A\mathbf{x} \approx \mathbf{b}$ where

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 4 \\ 0 & 3 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ -1 \end{bmatrix}.$$

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2a

Find a reduced QR factorization of A, i.e., $A = Q_1R_1$ for $Q_1 \in \mathbb{R}^{3,2}$ and $R_1 \in \mathbb{R}^{2,2}$, where Q has orthonormal columns and R_1 is upper triangular.

Answer: We apply Gram-Schmidt orthonormalization to the columns $\mathbf{a}_1, \mathbf{a}_2$ of A.

Step 1. Let $\mathbf{w} = \mathbf{a}_1$. Then

$$\mathbf{q}_1 = \frac{\mathbf{w}}{\|\mathbf{w}\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and $R_{11} = \|\mathbf{w}\| = 3$.

Step 2. Let $R_{12} = \langle \mathbf{q}_1, \mathbf{a}_2 \rangle = -1$ and

$$\mathbf{w} = \mathbf{a}_2 - R_{2,1}\mathbf{q}_1 = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}.$$

Then let

$$\mathbf{q}_2 = \frac{\mathbf{w}}{\|\mathbf{w}\|} = \begin{bmatrix} 0 \\ 4/5 \\ 3/5 \end{bmatrix}$$

and $R_{22} = \|\mathbf{w}\| = 5$.

Thus,

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 4/5 \\ 0 & 3/5 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix}.$$

2b

Using Q_1 and R_1 find the solution $\mathbf{x}$.

Answer: The solution is found by solving

$$R_1\mathbf{x} = Q_1^T\mathbf{b},$$

i.e.,

$$\begin{aligned} 3x_1 - x_2 &= 5, \\ 5x_2 &= 5, \end{aligned}$$

and so $x_2 = 1$ and $x_1 = 2$.

Problem 3 SVD

Consider the matrix

$$A = \begin{bmatrix} 1 & 8/3 \\ 0 & 1 \end{bmatrix}.$$

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3a

Given that one eigenpair $(\lambda_1, \mathbf{v}_1)$ of $A^T A$ is

$$\lambda_1 = 9, \quad \mathbf{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

find the other eigenpair $(\lambda_2, \mathbf{v}_2)$, with $\mathbf{v}_2$ normalized.

Answer:

$$B := A^T A = \begin{bmatrix} 1 & 0 \\ 8/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 8/3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 8/3 \\ 8/3 & 73/9 \end{bmatrix}.$$

Its eigenvalues satisfy the equation

$$\det(B - \lambda I) = 0,$$

i.e.,

$$\lambda^2 - \frac{82}{9}\lambda + 1 = 0,$$

whose roots are $\lambda_1 = 9$ and $\lambda_2 = 1/9$. We are given $\mathbf{v}_1$. If the other eigenvector is $\mathbf{v}_2 = [x, y]^T$ then

$$B \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{9} \begin{bmatrix} x \\ y \end{bmatrix}.$$

So we can take $x = 3$ and $y = -1$, and normalizing gives

$$\mathbf{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

3b

By letting

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \text{and} \quad U = AVD^{-1/2}$$

(or otherwise), find an SVD of A.

Answer: We have

$$\Sigma = D^{1/2} = \begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix},$$

and

$$D^{-1/2} = \begin{bmatrix} 1/3 & 0 \\ 0 & 3 \end{bmatrix},$$

Then

$$VD^{-1/2} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 3 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1/3 & 9 \\ 1 & -3 \end{bmatrix}.$$

Therefore,

$$U = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 8/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 9 \\ 1 & -3 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}.$$

The SVD of A is $A = U\Sigma V^T$ and its singular values are $\sigma_1 = 3$, $\sigma_2 = 1/3$.

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Problem 4 Interpolation

Let f be a function with four continuous derivatives in the interval $I = [-2h, 2h]$, for some $h > 0$. Let $p(x)$ be the polynomial of degree ≤ 3 such that $p(jh) = f(jh)$ for $j = -2, -1, 1, 2$. Show that for $x \in [-h, h]$,

$$|f(x) - p(x)| \leq \frac{1}{6} h^4 M_4, \quad \text{where} \quad M_4 = \max_{x \in I} |f^{(4)}(x)|.$$

Answer:

The interpolation error is given by

$$e(x) = f(x) - p(x) = \omega(x) \frac{f^{(4)}(\xi)}{4!},$$

where

$$\begin{aligned} \omega(x) &= (x + 2h)(x + h)(x - h)(x - 2h) \\ &= (x^2 - 4h^2)(x^2 - h^2) \\ &= x^4 - 5h^2 x^2 + 4h^4. \end{aligned}$$

If $x \in [-h, h]$ then $\xi \in I$, and so

$$|f(x) - p(x)| \leq \frac{1}{24} \max_{x \in [-h,h]} |\omega(x)| M_4. \tag{1}$$

The maximum of $|\omega(x)|$ is attained either at the end points of the interval $[-h, h]$ or at a point $x \in (-h, h)$ where $\omega'(x) = 0$. We have $\omega'(x) = 4x^3 - 10h^2 x$. The only point in $(-h, h)$ where $\omega'(x) = 0$ is $x = 0$. This implies

$$\max_{x \in [-h,h]} |\omega(x)| = \max\{|\omega(-h)|, |\omega(0)|, |\omega(h)|\} = |\omega(0)| = 4h^4.$$

Putting this into (1), we obtain

$$|f(x) - p(x)| \leq \frac{1}{6} h^4 M_4.$$

Problem 5 Bernstein basis

A Bézier curve of degree n in $\mathbb{R}^2$ is a parametric curve $\mathbf{p}(t) = \sum_{i=0}^n B_{i,n}(t) \mathbf{c}_i$ with control points $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{R}^2$ and Bernstein basis functions

$$B_{i,n}(t) = \binom{n}{i} t^i (1 - t)^{n-i},$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$. The first derivative $\mathbf{p}'(t)$ can be expressed as a Bézier curve of degree $n - 1$. What are its control points? Explain your answer.

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Answer: The derivative of $\mathbf{p}$ is

$$\mathbf{p}'(t) = \sum_{i=0}^{n} B'_{i,n}(t)\mathbf{c}_i,$$

and we can express $B'_{i,n}(t)$ in terms of Bernstein polynomials of lower degree. By the product rule,

$$\begin{aligned} B'_{i,n}(t) &= \frac{n!}{i!(n-i)!}\left(it^{i-1}(1-t)^{n-i} - t^i(n-i)(1-t)^{n-i-1}\right) \\ &= \frac{n!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{n!}{i!(n-i-1)!}t^i(1-t)^{n-i-1} \\ &= n(B_{i-1,n-1}(t) - B_{i,n-1}(t)), \end{aligned}$$

where $B_{-1,n-1}$ and $B_{n,n-1}$ are defined to be zero. Therefore,

$$\begin{aligned} \mathbf{p}'(t) &= n\sum_{i=0}^{n}(B_{i-1,n}(t) - B_{i,n-1}(t))\mathbf{c}_i \\ &= n\left(\sum_{i=1}^{n} B_{i-1,n}(t)\mathbf{c}_i - \sum_{i=0}^{n-1} B_{i,n}(t)\mathbf{c}_i\right) \\ &= n\left(\sum_{i=0}^{n-1} B_{i,n}(t)\mathbf{c}_{i+1} - \sum_{i=1}^{n-1} B_{i,n}(t)\mathbf{c}_i\right) \\ &= n\sum_{i=0}^{n-1} B_{i,n}(t)(\mathbf{c}_{i+1} - \mathbf{c}_i), \end{aligned}$$

and so the control points are $n(\mathbf{c}_{i+1} - \mathbf{c}_i)$, $i = 0, 1, \ldots, n-1$.

Problem 6 Multivariate Newton's method

6a

Write down Newton's method for minimizing a function $f(\mathbf{x})$, where $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$.

Answer: Newton's method is the root-finding method applied to the gradient $\nabla f(\mathbf{x})$, which is

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - [Hf(\mathbf{x}^{(k)})]^{-1}\nabla f(\mathbf{x}^{(k)}), \quad k = 0, 1, 2, \ldots,$$

with $\mathbf{x}^{(0)} \in \mathbb{R}^2$ an initial guess.

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6b

Suppose $f(\mathbf{x}) = x^4 + 2x^2y^2 + y^4$. If $\mathbf{x} = (a,a)$ for some $a \in \mathbb{R}$, show that the gradient and Hessian of f at $\mathbf{x}$ are

$$\nabla f(\mathbf{x}) = 8a^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad Hf(\mathbf{x}) = 8a^2 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Answer: We have

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 4x^3 + 4xy^2 \\ 4x^2y + 4y^3 \end{bmatrix},$$

and

$$Hf(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 12x^2 + 4y^2 & 8xy \\ 8xy & 4x^2 + 12y^2 \end{bmatrix}.$$

Letting $(x,y) = (a,a)$ gives the claimed formulas.

6c

If the start vector is $\mathbf{x}^{(0)} = (a,a)$ for some $a \neq 0$, find the next iterate, $\mathbf{x}^{(1)}$, and deduce the rate of convergence to the global minimizer $(0,0)$ in this case.

Answer: Since

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

we find

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - \frac{1}{24a^2} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} 8a^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{x}^{(0)} - \frac{a}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{3}\mathbf{x}^{(0)}.$$

This formula repeats,

$$\mathbf{x}^{(k+1)} = \frac{2}{3}\mathbf{x}^{(k)}.$$

Since the solution is $\mathbf{x}_* = (0,0)$, the k-th error $\mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}_*$ satisfies the same equation,

$$\mathbf{e}^{(k+1)} = \frac{2}{3}\mathbf{e}^{(k)}.$$

Thus the method converges, but only linearly.

Good luck!