UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3110/MAT4110 — Introduction to numerical analysis

Day of examination: 26 November 2020

Examination hours: 15:00 – 19:00

This problem set consists of 4 pages.

Appendices: None

Permitted aids: All written aids

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note:

- There are in total 14 subproblems (1a, 1b, 2a, etc.), and you can get between 5 and 10 points for each sub-problem.
- All answers must be justified.

Problem 1 QR factorization

Let $\delta > 0$ and let

$$A = \begin{pmatrix} 3 & 3 & 0 \\ 0 & \delta & 1 \\ 4 & 4 & 0 \end{pmatrix}.$$

1a

Compute the QR factorization of A using the Gram–Schmidt algorithm.

1b

If δ is very small, what can go wrong if we run this algorithm on a computer? Be as specific as you can.

Problem 2

Let $A \in \mathbb{R}^{n\times n}$ be a given matrix and define

$$f(x) = \frac{\|Ax\|}{\|x\|} \qquad \text{for } x \in \mathbb{R}^n,\ x \neq 0$$

(where $\|\cdot\|$ is the Euclidean norm, $\|x\| = \sqrt{\sum_{i=1}^n (x_i)^2}$.)

2a

In what way does $f(x)$ tell us how sensitive A is to x?

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2b

Assume that $n = 2$ and that A can be decomposed as

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{pmatrix}.$$

What type of decomposition is this?

2c

Let A be as in problem **2b**. Find nonzero vectors $y, z \in \mathbb{R}^2$ such that

$$f(y) = \max_{\substack{x \in \mathbb{R}^2 \\ x \neq 0}} f(x), \qquad f(z) = \min_{\substack{x \in \mathbb{R}^2 \\ x \neq 0}} f(x).$$

Problem 3 Newton's method

Consider the system of equations

$$\begin{aligned} x^3 - y^3 &= 3 \\ x^2 + y^2 &= 4. \end{aligned} \tag{1}$$

3a

Write down Newton's method for solving (1), and perform one iteration when starting at $(x_0, y_0) = (1, 1)$.

3b

What can go wrong if we start at, or close to, the x- or y-axes?

Problem 4 Polynomial interpolation

Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(0) = 2, \quad f(1) = 1, \quad f(3) = 5.$$

4a

Find the polynomial p of the lowest possible order which interpolates f through these points.

4b

Assume that $f \in C^3([0, 4])$ satisfies

$$\|f^{(3)}\|_\infty \leqslant M, \qquad \text{where } \left\|f^{(3)}\right\|_\infty = \sup_{x \in [0,4]} \left|f^{(3)}(x)\right|$$

for some $M > 0$. Estimate the error $\|f - p\|_\infty$ in terms of M.

*Hint: You don't need the solution from problem **4a** in order to solve problem **4b**.*

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Problem 5 Numerical quadrature

Consider the quadrature rule $I(f) \approx J(f)$, where

$$I(f) = \int_0^4 f(x)\,dx \quad \text{and} \quad J(f) = f(0)w_0 + f(x_1)w_1 + f(4)w_2$$

where $w_0, w_1, w_2 \in \mathbb{R}$ and $x_1 \in (0,4)$.

5a

Let $x_1 \in (0,4)$ be fixed. Let m denote the largest integer such that the quadrature rule is exact for all $f \in \mathbb{P}_m$. Show that w_0, w_1, w_2 can be chosen such that $m \geqslant 2$.

Hint: You need to find expressions for w_0, w_1, w_2, and to show that $I(f) = J(f)$ for all $f \in \mathbb{P}_2$.

5b

Is there a choice of $x_1 \in (0,4)$ which makes the order m of the quadrature rule greater than 2? Explain why, or why not.

Problem 6 Numerical methods for ODEs

Consider the ordinary differential equation (ODE)

$$\begin{cases} \dot{x}(t) = f(x(t),t) & \text{for } t > 0 \\ x(0) = x_0 \end{cases} \tag{2}$$

We partition the time domain $t \in [0,T]$ into subintervals with endpoints $t_n = hn$, where $h = \frac{T}{N}$, for some $N \in \mathbb{N}$. The solution is approximated as $y_n \approx x(t_n)$.

6a

Consider the Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|cc} \alpha & 0 & 0 \\ 2/3 & 1/3 & \beta \\ \hline & 1/4 & \gamma \end{array}$$

for numbers $\alpha, \beta, \gamma \in \mathbb{R}$. Determine α, β, γ such that the method is consistent. Is the method explicit or implicit?

6b

For the Runge–Kutta method from problem **6a**, write down a formula for y_{n+1} in terms of y_n.

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6c

We consider the linear problem (2) with $f(x,t) = \lambda x$ for some $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$. We solve the problem using the numerical method

$$\begin{cases} y_{n+1} = y_n + hf\big(y_n + \tfrac{h}{2}f(y_n,t_n),\ t_n + \tfrac{h}{2}\big) & \text{for } n = 0,1,\dots, \\ y_0 = x_0. \end{cases} \tag{3}$$

Find the stability function for (3). Is the method unconditionally stable?