UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3110/MAT4110 — Introduction to numerical analysis

Day of examination: 19 January 2021

Examination hours: 09:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: All written aids

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note:

- There are in total 11 subproblems (1, 2a, 2b, ...), and you can get 5–10 points for each sub-problem, for a total of 100 points.
- All answers must be justified.

Problem 1 Root finding

Let $f(x) = \cos(x) - x$. This function has a single root x_0 somewhere in $[0, 1]$, and we wish to compute it.

1a (10 points)

Perform two steps with both the bisection method and Newton's method. Justify your choice of starting values.

1b (10 points)

Which of the two methods can we expect to be the most accurate after several iterations? Justify your answer.

Problem 2 Polynomial interpolation (10 points)

Let $f : [0, 2] \to \mathbb{R}$ be a given function and let $n \in \mathbb{N}$. We wish to interpolate f using an n-th order polynomial p.

- Explain how we should do this in order to minimize the maximal error $\|f - p\|_{C([0,2])} = \sup_{x\in[0,2]} |f(x) - p(x)|$.
- Give an estimate of $\|f - p\|_{C([0,2])}$.

(Continued on page 2.)

Problem 3 Polynomial interpolation

Let $f : [0, 1] \to \mathbb{R}$ be the function $f(x) = \cos(2x) - e^x$. For some $n \in \mathbb{N}$, let p be the n-th order polynomial which interpolates f over the uniform grid $0, 1/n, \ldots, 1$.

3a (10 points)

Prove that $\|f - p\|_{C([0,1])} \to 0$ as $n \to \infty$.

(Here, $\|f - p\|_{C([0,1])} = \sup_{x\in[0,1]} |f(x) - p(x)|$.)

3b (10 points)

How large must n be in order to guarantee that $\|f - p\|_{C([0,1])} \leqslant 10^{-10}$?

Hint: In this problem you might (or might not) need Stirling's approximation:

$$m! \geqslant m^m e^{-m}.$$

Problem 4 QR factorization

Let A, Q and R be the matrices

$$A = \begin{pmatrix} 0 & 1 \\ \sqrt{2} & 3\sqrt{2} \\ 0 & 1 \end{pmatrix}, \quad R = \sqrt{2}\begin{pmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Note that $A = QR$ (you don't have to show this).

4a (5 points)

Explain what it means that QR is the QR factorization of A. Justify your answer.

4b (10 points)

Find the least squares solution of the equation

$$Ax = b, \qquad \text{where } b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Problem 5 SVD (10 points)

Compute the singular value decomposition (SVD) of

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}.$$

Hint: You may use the fact that one of the eigenpairs of the normal matrix $A^\mathsf{T} A$ *is* $\lambda_1 = 50$, $\mathbf{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(Continued on page 3.)

Problem 6

We wish to approximate the integral $I(f) = \int_0^{20} f(x)\,dx$ of a function f.

6a (5 points)

If we wish to approximate $I(f)$ using an 5-point quadrature rule, which quadrature rule should we choose to make the error as small as possible? Justify your answer.

6b (10 points)

Recall that the Gauss quadrature of order 3 on the interval $[-1, 1]$ is

$$\int_{-1}^{1} g(x)\,dx \approx f\big(-\sqrt{1/3}\big) + f\big(\sqrt{1/3}\big). \tag{1}$$

Write down the composite integration rule over $N = 2$ subintervals which approximates $I(f)$. Use the quadrature rule (1) in the composite method.

Problem 7 Runge–Kutta method (10 points)

Consider the ODE

$$\begin{cases} x'(t) = f(x(t), t) \\ x(0) = x_0 \end{cases}$$

where f is a given smooth function, and the Runge–Kutta method

$$\begin{aligned} k &= f(y_n + hk/2, t_n + h/2) \\ y_{n+1} &= y_n + hk. \end{aligned}$$

Set $f(x, t) = \lambda x$ for some $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) < 0$. Find the stability function of this method, and determine whether the method is unconditionally stable or not.

Hint: If you are unable to determine stability, it's enough to insert $h\lambda = 1, 10, 100$ *in the stability function and conclude based on that.*