UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3110/MAT4110 — Introduction to numerical analysis
Day of examination:	15 December 2020
Examination hours:	09:00 – 13:00

This problem set consists of 3 pages.

Appendices:	None
Permitted aids:	All written aids

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note:

- There are in total 12 subproblems (1a, 1b, and so on), and you can get 10 points for each sub-problem.
- All answers must be justified.

Problem 1 Condition number

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & \varepsilon \end{pmatrix}$ for some (small) number $\varepsilon > 0$.

1a

Show that the condition number $\kappa_\infty(A)$ with respect to the supremum norm $\|\cdot\|_\infty$ is $3 + \frac{6}{\varepsilon}$.

1b

Let $\varepsilon = 10^{-4}$. If we want to solve the equation $Ax = c$, and we make a relative error $\frac{\|\delta c\|_\infty}{\|c\|_\infty} = 0.001$ in c, then how large might the relative error (measured in the ∞-norm) in the solution x be?

Problem 2 Solving nonlinear equations

Let $f(x) = \begin{pmatrix} (1 - a^2 b)/4 \\ (a^2 + b^2 + 1)/8 \end{pmatrix}$ for $x = (a, b) \in D = [0, 1]^2$. We wish to solve the fixed point equation

$$f(x) = x. \tag{1}$$

We consider a fixed point iteration starting at $x^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2a

Compute the first iteration $x^{(1)}$ of the fixed point iteration for this problem.

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2b

Show that f is a contraction in the norm $\|\cdot\|_\infty$ with contraction constant $L = 3/4$. Prove that the fixed point iteration converges to a solution of (1).

2c

Approximately how many fixed point iterations are needed when starting at $x^{(0)}$, in order to guarantee that the error $\|x^{(k)} - x\|_\infty$ is less than 10^{-3}?

Problem 3 Interpolation

Let $f(x) = \frac{1}{1+x}$ for $x \in [0,1]$.

3a

Let $p_n \in \mathcal{P}_n$ be the interpolant of f over the uniform mesh $x_k = \frac{k}{n}$, $k = 0, 1, \ldots, n$. Estimate the error $\|f - p_n\|_\infty$. Is it possible to find an $n \in \mathbb{N}$ so that the error is less than 10^{-4}?

Hint: The j-th derivative of f is $f^{(j)}(x) = \frac{(-1)^j j!}{(1+x)^{j+1}}$.

3b

For a fixed $n \in \mathbb{N}$, how should we select the interpolation points $x_0, \ldots, x_n \in [0,1]$ in order to make the error $\|f - p_n\|_\infty$ as small as possible? Estimate the error for these interpolation points.

Problem 4 Approximation in the 2-norm

Define the weight function $w\colon [0,1] \to \mathbb{R}$ by $w(x) = x$. Find the polynomial $p \in \mathcal{P}_1$ which is closest to $f(x) = e^x$ in the weighted 2-norm

$$\|f - p\|_{L^2_w} = \sqrt{\int_0^1 w(x)|f(x) - p(x)|^2\,dx}.$$

Hint: The first orthogonal polynomials with respect to w are

$$\varphi_0(x) = 1, \qquad \varphi_1(x) = x - \frac{2}{3}$$

(you don't need to show this).

Problem 5 Order of a quadrature rule

We want to approximate the integral $I(f) = \int_{-1}^1 f(x)\,dx$. Let $x_0 = -\frac{2}{3}$. Find $x_1 \in [-1,1]$ so that the resulting quadrature method

$$I(f) \approx I_1(f) = w_0 f(x_0) + w_1 f(x_1)$$

has order at least 3. Is it possible to find x_1 so that the order is 4?

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Problem 6 Composite quadrature

We wish to approximate the d-dimensional integral

$$I(f) \coloneqq \int_{[0,1]^d} f(x)\,dx$$

for $f\colon [0,1]^d \to \mathbb{R}$. Consider the midpoint method

$$I(f) \approx I_0(f) \coloneqq f(1/2, 1/2, \dots, 1/2).$$

6a

Prove that the approximation error can be bounded by

$$|I(f) - I_0(f)| \leqslant \frac{1}{8}\big\|\nabla^2 f\big\|_\infty$$

where $\nabla^2 f$ is the Hessian of f and the supremum is taken with respect to the matrix norm: $\|\nabla^2 f\|_\infty = \sup_{x\in[0,1]^d} \|\nabla^2 f(x)\|_{\mathcal{L}}$.

Hint: You might need the multi-dimensional Taylor expansion,

$$f(x) = f(a) + \nabla f(a)\cdot(x-a) + R(x)$$

where the remainder term can be bounded as $|R(x)| \leqslant \frac{1}{2}\big\|\nabla^2 f\big\|_\infty \|x-a\|_\infty^2$.

6b

Write down the composite midpoint method $I_{0,m}$ for the above integral, and show that the error is at most

$$|I(f) - I_{0,m}(f)| \leqslant \frac{\|\nabla^2 f\|_\infty}{8m^2}.$$

6c

If $d = 20$ and $\|\nabla^2 f\|_\infty \approx 1$, *roughly* how many function evaluations of f are needed in order to bring the error below $\varepsilon = 10^{-4}$? Is this feasible (realistic) on a modern laptop?