UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3110/MAT4110 — Introduction to numerical analysis

Day of examination: 15 December 2020

Examination hours: 09:00 – 13:00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: All written aids

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note:

- There are in total 12 subproblems (1a, 1b, and so on), and you can get 10 points for each sub-problem.
- All answers must be justified.

Problem 1 Condition number

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & \varepsilon \end{pmatrix}$ for some (small) number $\varepsilon > 0$.

1a

Show that the condition number $\kappa_\infty(A)$ with respect to the supremum norm $\|\cdot\|_\infty$ is $3 + \frac{6}{\varepsilon}$.

1b

Let $\varepsilon = 10^{-4}$. If we want to solve the equation $Ax = c$, and we make a relative error $\frac{\|\delta c\|_\infty}{\|c\|_\infty} = 0.001$ in c, then how large might the relative error (measured in the ∞-norm) in the solution x be?

Solution:

1a

The matrix norm of A is the largest (absolute) row-sum of A. Hence,

$$\|A\|_{\mathcal{L}} = 1 + 2 = 3, \qquad \|A^{-1}\|_{\mathcal{L}} = \left\| \begin{pmatrix} 1 & -2/\varepsilon \\ 0 & 1/\varepsilon \end{pmatrix} \right\|_{\mathcal{L}} = 1 + \frac{2}{\varepsilon}.$$

Hence, $\kappa_\infty(A) = \|A\|_{\mathcal{L}} \|A^{-1}\|_{\mathcal{L}} = 3 + \frac{6}{\varepsilon}$.

(Continued on page 2.)

1b

The relative error in x might be as large as

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leqslant \kappa_\infty(A)\frac{\|\delta c\|_\infty}{\|c\|_\infty} = \left(3 + \tfrac{6}{\varepsilon}\right)10^{-3} \approx 60.$$

Problem 2 Solving nonlinear equations

Let $f(x) = \begin{pmatrix} (1-a^2b)/4 \\ (a^2+b^2+1)/8 \end{pmatrix}$ for $x = (a,b) \in D = [0,1]^2$. We wish to solve the fixed point equation

$$f(x) = x. \tag{1}$$

We consider a fixed point iteration starting at $x^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

2a

Compute the first iteration $x^{(1)}$ of the fixed point iteration for this problem.

2b

Show that f is a contraction in the norm $\|\cdot\|_\infty$ with contraction constant $L = 3/4$. Prove that the fixed point iteration converges to a solution of (1).

2c

Approximately how many fixed point iterations are needed when starting at $x^{(0)}$, in order to guarantee that the error $\|x^{(k)} - x\|_\infty$ is less than 10^{-3}?

Solution:

2a

We get $x^{(1)} = f(x^{(0)}) = \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}$.

2b

The Jacobian of f is

$$J_f(x) = \begin{pmatrix} -ab/2 & -a^2/4 \\ a/4 & b/4 \end{pmatrix}.$$

The matrix norm of the Jacobian is

$$\|J_f(x)\|_{\mathcal{L}} = \max(ab/2 + a^2/4, a/4 + b/4) \leqslant 3/4.$$

Since D is convex, it follows that f is a contraction with contraction constant $L = 3/4$.

We also note that f maps D into D: The first component always lies in $[1/4, 1/2]$, and the second component always lies in $[1/8, 3/8]$. By

(Continued on page 3.)

Banach's fixed point theorem, the fixed point iteration converges to a solution of (1).

2c

We have the basic estimate

$$\|x^{(k)} - x\| \leqslant \|x^{(1)} - x^{(0)}\| \frac{L^k}{1-L} = \frac{3}{4}\frac{(3/4)^k}{1/4} = 3(3/4)^k.$$

Thus, the error is less than 10^{-3} if

$$k > \frac{\log(1/3 \cdot 10^{-3})}{\log(3/4)} \approx 27.8.$$

Hence, we need $k \geqslant 28$.

Problem 3 Interpolation

Let $f(x) = \frac{1}{1+x}$ for $x \in [0,1]$.

3a

Let $p_n \in \mathcal{P}_n$ be the interpolant of f over the uniform mesh $x_k = \frac{k}{n}$, $k = 0, 1, \ldots, n$. Estimate the error $\|f - p_n\|_\infty$. Is it possible to find an $n \in \mathbb{N}$ so that the error is less than 10^{-4}?

Hint: The j-th derivative of f is $f^{(j)}(x) = \frac{(-1)^j j!}{(1+x)^{j+1}}$.

3b

For a fixed $n \in \mathbb{N}$, how should we select the interpolation points $x_0, \ldots, x_n \in [0,1]$ in order to make the error $\|f - p_n\|_\infty$ as small as possible? Estimate the error for these interpolation points.

Solution:

3a

We have $\|f - p_n\|_\infty \leqslant \frac{\|f^{(n+1)}\|}{(n+1)!}\|\pi_{n+1}\|_\infty = \|\pi_{n+1}\|_\infty = \sup_{x\in[0,1]} |x - 0||x - 1/n| \cdots |x - 1| \leqslant 1/n \to 0$ as $n \to \infty$. Hence, the answer is "yes".

3b

The polynomial minimizing $\|f - p\|_\infty$ is the minimax polynomial, but we have no general approach to finding this polynomial.

Chebyshev interpolation points would yield the smallest value of $\|\pi_{n+1}\|_\infty$ for a general function f. If T_{n+1} is the Chebyshev polynomial on $[-1,1]$ then $C_{n+1}(x) = T_{n+1}(2x-1)$ is the Chebyshev polynomial on $[0,1]$, and $2^{-n-1}2^{-n}C_{n+1} = 2^{-2n-1}C_{n+1}$ is a monic polynomial with

(Continued on page 4.)

zeros $x_0, \dots, x_n$, and hence equals π_{n+1}. We get

$$\|f - p_n\|_\infty \leqslant \|\pi_{n+1}\|_\infty = 2^{-2n-1}\|C_{n+1}\|_\infty = 2^{-2n-1}.$$

Problem 4 Approximation in the 2-norm

Define the weight function $w\colon [0,1] \to \mathbb{R}$ by $w(x) = x$. Find the polynomial $p \in \mathcal{P}_1$ which is closest to $f(x) = e^x$ in the weighted 2-norm

$$\|f - p\|_{L^2_w} = \sqrt{\int_0^1 w(x)|f(x) - p(x)|^2\,dx}.$$

Hint: The first orthogonal polynomials with respect to w are

$$\varphi_0(x) = 1, \qquad \varphi_1(x) = x - \frac{2}{3}$$

(you don't need to show this).

Solution: We have the projection formula

$$p = \sum_{i=0}^{1} \frac{\langle \varphi_i, f\rangle}{\langle \varphi_i, \varphi_i\rangle}\varphi_i.$$

We compute

$$\langle \varphi_0, f\rangle = \int_0^1 xe^x = 1, \qquad \langle \varphi_0, \varphi_0\rangle = \frac{1}{2},$$
$$\langle \varphi_1, f\rangle = \int_0^1 x(x - \tfrac{2}{3})e^x = e - \frac{8}{3} \qquad \langle \varphi_1, \varphi_1\rangle = \frac{1}{36}.$$

Hence,

$$p(x) = 2 + 36\big(e - \tfrac{8}{3}\big)\big(x - \tfrac{2}{3}\big).$$

Problem 5 Order of a quadrature rule

We want to approximate the integral $I(f) = \int_{-1}^{1} f(x)\,dx$. Let $x_0 = -\frac{2}{3}$. Find $x_1 \in [-1,1]$ so that the resulting quadrature method

$$I(f) \approx I_1(f) = w_0 f(x_0) + w_1 f(x_1)$$

has order at least 3. Is it possible to find x_1 so that the order is 4?

Solution: The Lagrange functions are $L_0(x) = \frac{x-x_0}{x_1-x_0}$ and $L_1(x) = \frac{x_1-x}{x_1-x_0}$, so the weights are $w_0 = \frac{-2x_0}{x_1-x_0}$ and $w_1 = \frac{2x_1}{x_1-x_0}$.

A quadrature method over two quadrature points has order at least 2 and at most 4, so we only need to check that the order is at least 3, that is, that the polynomial $p_2(x) = x^2$ is integrated exactly. On the one hand, $I(p_2) = \frac{2}{3}$. On the other hand,

$$I_1(p_2) = w_0x_0^2 + w_1x_1^2 = \frac{-2x_1x_0^2}{x_1 - x_0} + \frac{2x_0x_1^2}{x_1 - x_0} = -2x_0x_1.$$

(Continued on page 5.)

We want this to be equal to $I(p_2) = \frac{2}{3}$, so we need

$$x_1 = -\frac{1}{2x_0}\frac{2}{3} = \frac{1}{2}.$$

The only quadrature rule of order 4 is the Gauss method, and since the quadrature point $-\frac{2}{3}$ is not one of the Gauss quadrature points, we cannot achieve 4th order.

Problem 6 Composite quadrature

We wish to approximate the d-dimensional integral

$$I(f) \coloneqq \int_{[0,1]^d} f(x)\,dx$$

for $f\colon [0,1]^d \to \mathbb{R}$. Consider the midpoint method

$$I(f) \approx I_0(f) \coloneqq f(1/2, 1/2, \ldots, 1/2).$$

6a

Prove that the approximation error can be bounded by

$$|I(f) - I_0(f)| \leqslant \frac{1}{8}\left\|\nabla^2 f\right\|_\infty$$

where $\nabla^2 f$ is the Hessian of f and the supremum is taken with respect to the matrix norm: $\|\nabla^2 f\|_\infty = \sup_{x\in[0,1]^d} \|\nabla^2 f(x)\|_{\mathcal{L}}$.

Hint: You might need the multi-dimensional Taylor expansion,

$$f(x) = f(a) + \nabla f(a)\cdot(x-a) + R(x)$$

where the remainder term can be bounded as $|R(x)| \leqslant \frac{1}{2}\left\|\nabla^2 f\right\|_\infty \|x-a\|_\infty^2$.

6b

Write down the composite midpoint method $I_{0,m}$ for the above integral, and show that the error is at most

$$|I(f) - I_{0,m}(f)| \leqslant \frac{\|\nabla^2 f\|_\infty}{8m^2}.$$

6c

If $d = 20$ and $\|\nabla^2 f\|_\infty \approx 1$, *roughly* how many function evaluations of f are needed in order to bring the error below $\varepsilon = 10^{-4}$? Is this feasible (realistic) on a modern laptop?

Solution:

6a

Set $a = (1/2, \ldots, 1/2)$ in the first-order Taylor expansion of f. Then

$$I(f) = \int_{[0,1]^d} f(a) + \nabla f(a)\cdot(x-a) + R(x)\,dx$$

(Continued on page 6.)

(set $\overline{R} = \int_{[0,1]^d} R(x)\,dx$)

$$
\begin{aligned}
&= f(a) + \nabla f(a) \cdot \int_{[0,1]^d} x - a\,dx + \overline{R} \\
&= I_0(f) + \overline{R}
\end{aligned}
$$

where we used the fact that $I_0(f) = f(a)$ and that $\int x - a\,dx = 0$. We can estimate $\overline{R}$ by

$$|\overline{R}| \leqslant \int_{[0,1]^d} \frac{1}{2}\|\nabla^2 f\|_\infty \|x-a\|_\infty^2\,dx \leqslant \frac{1}{8}\|\nabla^2 f\|_\infty$$

since $\|x-a\|_\infty \leqslant 1/2$ for all x in the unit hypercube.

6b

The composite midpoint method is

$$I_{0,m}(f) = \frac{1}{m^d}\sum_{i_1=1}^{m}\cdots\sum_{i_d=1}^{m} f\Big(\tfrac{i_1-1/2}{m},\dots,\tfrac{i_d-1/2}{m}\Big).$$

In each hypercube $\mathcal{C}_i = \big[\frac{i_1-1}{m},\frac{i_1}{m},\dots,\frac{i_d-1}{m},\frac{i_d}{m}\big]$, the error committed can be bounded by

$$
\begin{aligned}
\left|\int_{\mathcal{C}_i} f(x)\,dx - \tfrac{1}{m^d} f\Big(\tfrac{i_1-1/2}{m},\dots,\tfrac{i_d-1/2}{m}\Big)\right| &= \left|\int_{[0,1]^d} \tilde{f}(y)\,dy - \tilde{f}(1/2,\dots,1/2)\right| \\
&\leqslant \frac{1}{8}\|\nabla^2 \tilde{f}\|_\infty
\end{aligned}
$$

where $\tilde{f}(y) = \frac{1}{m^d} f\Big(\frac{i_1-y}{m},\dots,\frac{i_d-y}{m}\Big)$. We have $\|\nabla^2\tilde{f}\|_\infty \leqslant \frac{1}{m^{d+2}}\|\nabla^2 f\|_\infty$, so summing up all the errors gives

$$|I_{0,m} - I(f)| \leqslant \sum_{i_1=1}^{m}\cdots\sum_{i_d=1}^{m}\frac{1}{8}\frac{1}{m^{d+2}}\|\nabla^2 f\|_\infty = \frac{1}{8m^2}\|\nabla^2 f\|_\infty.$$

6c

If $\frac{1}{8m^2}\|\nabla^2 f\|_\infty < \varepsilon$, i.e. $m > \sqrt{\frac{\|\nabla^2 f\|_\infty}{8\varepsilon}} \approx \sqrt{\frac{1}{8\varepsilon}}$, then the error is no bigger than ε. Since the quadrature method uses $m_{\text{tot}} = m^d$ function evaluations, this corresponds to

$$m_{\text{tot}} > \frac{1}{8^{d/2}\varepsilon^{d/2}} = 8^{-10}10^{40} \approx 10^{31}.$$

The number of calculations needed is prohibitively large. (For comparison, today's fastest supercomputers can perform about 10^{18} floating point operations per second.)