UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	MAT-INF 4130 — Numerical linear algebra
Day of examination:	3 December 2013
Examination hours:	1100 – 1500

This problem set consists of 2 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 9 part questions will be weighted equally.

Problem 1 True or false

Give reasons for your answers.

1a

If two matrices have the same eigenvalues they must be similar.

1b

If $\boldsymbol{x} \in \operatorname{span}(\boldsymbol{A})$ and $\boldsymbol{y} \in \ker(\boldsymbol{A})$ then $\boldsymbol{x}^T\boldsymbol{y} = 0$ for any $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$.

1c

The overdetermined linear system

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix} = \boldsymbol{b}$$

has a least squares solution $x_1 = -6$, $x_2 = 9/2$. This solution is unique.

1d

The matrix

$$\boldsymbol{A} := \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & 5 & -10 \\ 0 & 9 & 0 & 10 \end{bmatrix}$$

(Continued on page 2.)

has a unique LU-factorization. (Do not compute the factorization.)

Problem 2 Givens rotation

A Givens rotation of order 2 has the form $\boldsymbol{G} := \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \in \mathbb{R}^{2\times 2}$, where $s^2 + c^2 = 1$.

2a

Is $\boldsymbol{G}$ symmetric and unitary?

2b

Given $x_1, x_2 \in \mathbb{R}$ and set $r := \sqrt{x_1^2 + x_2^2}$. Find $\boldsymbol{G}$ and y_1, y_2 so that $y_1 = y_2$, where $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boldsymbol{G} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Problem 3 Perturbation of the identity matrix

Let $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ and suppose $\|\boldsymbol{B}\| < 1$ for some operator norm.

3a

Show that $\boldsymbol{I} - \boldsymbol{B}$ is nonsingular.

3b

Show that

$$\|(\boldsymbol{I} - \boldsymbol{B})^{-1}\| \le \frac{1}{1 - \|\boldsymbol{B}\|}.$$

Problem 4 Matlab program

Suppose $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{b} \in \mathbb{R}^m$, where $\boldsymbol{A}$ has rank n and let $\boldsymbol{A} = \boldsymbol{U\Sigma V}^T$ be a singular value factorization of $\boldsymbol{A}$. Thus $\boldsymbol{U} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{\Sigma}, \boldsymbol{V} \in \mathbb{R}^{n\times n}$. Write a Matlab `function [x,K]=lsq(A,b)` that uses the singular value factorization of $\boldsymbol{A}$ to calculate a least squares solution $\boldsymbol{x} = \boldsymbol{V\Sigma}^{-1}\boldsymbol{U}^T\boldsymbol{b}$ to the system $\boldsymbol{Ax} = \boldsymbol{b}$ and the spectral (2-norm) condition number of $\boldsymbol{A}$. The Matlab command `[U,Sigma,V]=svd(A,0)` computes the singular value factorization of $\boldsymbol{A}$.

Good luck!