Mat 3110 Monte Carlo integration – volume of unit ball

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Volume of d-dimensional unit ball

Let $B_d = \{x \in \mathbb{R}^d \mid \|x\|_2 \leq 1\}$ and let $|B_d|$ denote volume of unit ball in dimension $d \geq 1$.

Then

$$|B_d| = \int_{[0,1]^d} \underbrace{2^d \mathbb{1}_{B_d}(x)}_{=:f(x)} \, dx = I(f)$$

and this can be shown to equal

$$|B_d| = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$$

where

$$\Gamma(d/2+1) := \int_0^\infty t^{d/2} e^{-t} \, dt,$$

with the recursive property $\Gamma(y+1) = y\Gamma(y)$ for all $y > 0$.

Monte Carlo

Monte Carlo approximation:

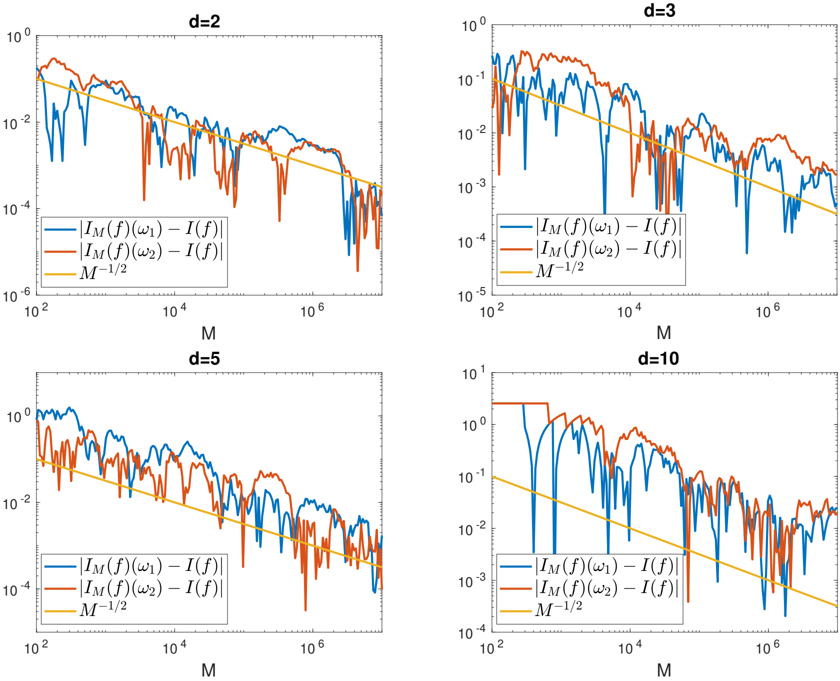
$$I_M(f) = \sum_{m=1}^{M} \frac{f(X_i)}{M}$$

where $X_1, \ldots, X_m \sim U([0,1]^d)$ are mutually independent.

By theory

$$\sqrt{\mathbb{E}\left[|I_M(f) - I(f)|^2\right]} \leq \frac{2^d}{\sqrt{M}},$$

for any fixed $d \geq 1$, do we observe this numerically?



d=2
d=3
d=5
d=10
$|I_M(f)(\omega_1) - I(f)|$
$|I_M(f)(\omega_2) - I(f)|$
$M^{-1/2}$
M

What happens here?