UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT3360 — Introduction to Partial Differential Equations

Day of examination: Thursday, June 7, 2018

Examination hours: 09:00 – 13:00

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let f be the signum function, i.e.,

$$f(x) = \operatorname{sign}(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

1a

Find the full Fourier series for f, S_f, on the interval $[-1, 1]$.

Possible solution: Since f is odd, we know that the a_k's all are zero. The b_k's are given by

$$b_k = \int_{-1}^{1} f(x) \sin(k\pi x)\, dx = 2 \int_0^1 \sin(k\pi x)\, dx = \begin{cases} \frac{4}{k\pi} & \text{if } k \text{ is odd}, \\ 0 & \text{if } k \text{ is even}. \end{cases}$$

Hence we get

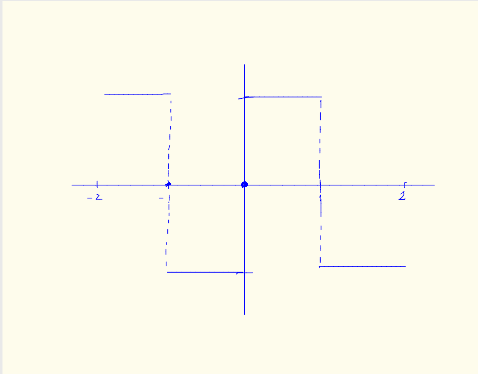
$$S_f(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)\pi x)}{2k-1}.$$

1b

Sketch the graph of $S_f(x)$ for $x \in (-2, 2)$.

Possible solution:

(Continued on page 2.)



1c

Use the previous results to show that

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \quad \text{and} \quad \frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

Possible solution: *There was a misprint here: "$(-1)^k$" should have been "$(-1)^{k+1}$"!!*
We have $S_f(1/2) = 1$, therefore

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{\sin((2k-1)\frac{\pi}{2})}{2k-1},$$

and the first identity follows since $\sin((2k-1)\frac{\pi}{2}) = (-1)^{k+1}$. Since $\|f\|^2 = 2$, the second identity is Parseval's identity for S_f.

Problem 2

Consider the following partial differential equation

$$\begin{cases} u_t + q(x)u_x = u_{xx}, & x \in (0,1), \quad t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = f(x). \end{cases} \tag{1}$$

Here q and f are continuous functions $[0,1] \to \mathbb{R}$,

2a

Show the maximum principle

$$\min\left\{0, \min_{x\in(0,1)} f(x)\right\} \le u(x,t) \le \max\left\{0, \max_{x\in(0,1)} f(x)\right\}.$$

(**Hint**: Consider $v = u - \varepsilon t$ and let $\varepsilon \downarrow 0$). Explain why this implies that (1) can have at most one solution.

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Possible solution: Note: the open interval $(0,1)$ should have been the closed interval $[0,1]$ in this question.
Set $v = u - \varepsilon t$, then $u_t = v_t + \varepsilon$, $u_x = v_x$ and $u_{xx} = v_{xx}$. Therefore

$$v_t + \varepsilon + q(x)v_x = v_{xx}.$$

Assume that v has a local maximum at $(\hat{x}, \hat{t})$ for $\hat{t} > 0$ and $\hat{x} \in (0,1)$. Then $v_t(\hat{x}, \hat{t}) \geq 0$, $v_x(\hat{x}, \hat{t}) = 0$ and $v_{xx}(\hat{x}, \hat{t}) \leq 0$.

$$0 \leq v_t(\hat{x}, \hat{t}) = v_t(\hat{x}, \hat{t}) + q(\hat{x})v_x(\hat{x}, \hat{t}) = v_{xx}(\hat{x}, \hat{t}) - \varepsilon \leq -\varepsilon.$$

This is a contradiction, and v cannot have local maxima, thus $v(x,t) \leq \max\left\{0, \max_{x\in[0,1]} f(x)\right\}$. Hence

$$u(x,t) \leq \max\left\{0, \max_{x\in[0,1]} f(x)\right\} + \varepsilon t,$$

for all positive ε. This implies the right inequality. To show the minimum principle, set $w = -u$ and apply the maximum principle.
Assuming we have two solutions, u and v, then $w = u - v$ will satisfy $w(x,0) = 0$, and hence $0 \leq w(x,t) \leq 0$.

2b

Define

$$E(t) = \frac{1}{2}\int_0^1 (u(x,t))^2 \, dx.$$

Show that

$$E'(t) = -\int_0^1 (u_x)^2 \, dx - \frac{1}{2}\int_0^1 q(x)(u^2)_x \, dx. \qquad (2)$$

Now we assume that q is continuously differentiable, such that $\|q'\|_\infty < \infty$, use (2) to establish the energy estimate

$$E(t) \leq E(0)e^{\|q'\|_\infty t}.$$

Possible solution: We get

$$\begin{aligned} E'(t) &= \int_0^1 u u_t \, dx = \int_0^1 u u_{xx} \, dx - \int_0^1 q(x) u u_x \, dx \\ &= \int_0^1 u u_{xx} \, dx - \int_0^1 q(x)\frac{1}{2}\left(u^2\right)_x \, dx \\ &= -\int_0^1 (u_x)^2 \, dx + \int_0^1 q'(x)\frac{1}{2}u^2 \, dx \\ &\leq \left\|q'\right\|_\infty E(t). \end{aligned}$$

Then Gronwall's inequality implies the energy bound.

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2c

Let $h : [0,1] \to \mathbb{R}$ be a continuously differentiable function such that $h(0) = h(1) = 0$. Show the inequality

$$(h(x))^2 \leq \min\{x, 1-x\} \int_0^1 \left(h'(y)\right)^2 \, dy \text{ for } x \in [0,1].$$

(**Hint**: Use that $|h(x)| = |\int_0^x h'(y)\,dy|$ and $|h(x)| = |\int_x^1 h'(y)\,dy|$ and the Cauchy-Schwartz inequality.)

Possible solution: We have

$$\begin{aligned}
|h(x)| &= \left|\int_0^x h'(y)\,dy\right| \qquad \text{use Cauchy Schwartz} \\
&\leq \sqrt{x}\Big(\int_0^x (h'(y))^2\,dy\Big)^{1/2} \\
&\leq \sqrt{x}\Big(\int_0^1 (h'(y))^2\,dy\Big)^{1/2}
\end{aligned}$$

We also have $|h(x)| = \left|\int_x^1 h'(y)\,dy\right|$ which gives us

$$|h(x)| \leq \sqrt{1-x}\Big(\int_0^1 (h'(y))^2\,dy\Big)^{1/2}.$$

2d

Show that $E(t) \leq E(0)$ if

$$\int_0^1 \min\{x, 1-x\} \left|q'(x)\right| \, dx \leq 2.$$

(**Hint**: Start with (2) and use **c**.)

Possible solution: We have

$$\begin{aligned}
E'(t) &= -\int_0^1 (u_x)^2 \, dx + \frac{1}{2}\int_0^1 q'(x)u^2\,dx \\
&\leq -\int_0^1 (u_x)^2 \, dx + \frac{1}{2}\int_0^1 (u_x)^2 \, dx \int_0^1 \min\{x, 1-x\}\left|q'(x)\right| \, dx \\
&= \frac{1}{2}\left(\int_0^1 \min\{x,1-x\}\left|q'(x)\right| \, dx - 2\right)\int_0^1 (u_x)^2\,dx \\
&\leq 0 \qquad \text{if the bound above holds.}
\end{aligned}$$

(Continued on page 5.)

Problem 3

Let $\Omega = \{(x,y) \mid x^2 + y^2 < 1\}$, and consider the following boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega, \\ u = g & (x,y) \in \partial\Omega, \end{cases} \tag{3}$$

where g is a given continuous function. In polar coordinates (r, φ) we have

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2},$$

you do not have to show this. Let $\Delta r = 1/(N + 1/2)$ and $\Delta\varphi = 2\pi/(M+1)$ for positive integers N and M, and set

$$r_i = (i - 1/2)\Delta r,\ i = 1, \ldots, N+1 \ \text{ and } \varphi_j = j\Delta\varphi,\ j = 0, \ldots M+1.$$

We are interested in finding $u_{ij} \approx u(r_i, \varphi_j)$.

3a

Explain why the following difference scheme is a reasonable approximation to (3).

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{(\Delta r)^2} + \frac{1}{r_i}\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta r} + \frac{1}{r_i^2}\frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{(\Delta\varphi)^2} = 0,$$
$$i = 1, \ldots N, \quad j = 1, \ldots, M,$$
$$u_{i,0} = u_{i,M},\ u_{i,1} = u_{i,M+1}, \quad i = 1, \ldots, N,$$
$$u_{N+1,j} = g(\varphi_j),\ j = 0, \ldots, M.$$

Possible solution: We have that

$$u_{rr} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{(\Delta r)^2} + \mathcal{O}((\Delta r)^2),$$
$$u_r = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta r} + \mathcal{O}((\Delta r)^2),$$
$$u_{\varphi\varphi} = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{(\Delta\varphi)^2} + \mathcal{O}((\Delta\varphi)^2)$$

The conditions $u_{i,0} = u_{i,M},\ u_{i,1} = u_{i,M+1}$ say that u_{ij} is periodic in j, while the conditions $u_{N+1,j} = g(\varphi_j)$ say that u takes the correct boundary values.

3b

Let m_{ij} and M_{ij} be defined as the minimum and the maximum of $u_{\cdot,\cdot}$ at the neighboring points of (r_i, φ_j), i.e.,

$$m_{ij} = \begin{cases} \min\{u_{i+1,j}, u_{i-1,j}, u_{i,j+1}, u_{i,j-1}\} & N \geq i > 1 \\ \min\{u_{2,j}, u_{1,j+1}, u_{1,j-1}\} & i = 1, \end{cases}$$
$$M_{ij} = \begin{cases} \max\{u_{i+1,j}, u_{i-1,j}, u_{i,j+1}, u_{i,j-1}\} & N \geq i > 1 \\ \max\{u_{2,j}, u_{1,j+1}, u_{1,j-1}\} & i = 1. \end{cases}$$

(Continued on page 6.)

Show the discrete maximum principle

$$m_{ij} \leq u_{ij} \leq M_{ij},\ i = 1, \ldots, N,\ j = 1, \ldots, M.$$

Possible solution: We solve for u_{ij}

$$\underbrace{\left(\frac{2}{(\Delta r)^2} + \frac{2}{r_i^2}\frac{1}{(\Delta\varphi)^2}\right)}_{\Gamma} u_{ij} = \underbrace{\left(\frac{1}{(\Delta r)^2} + \frac{1}{2r_i\Delta r}\right)}_{\alpha} u_{i+1,j} + \underbrace{\left(\frac{1}{(\Delta r)^2} - \frac{1}{2r_i\Delta r}\right)}_{\beta} u_{i-1,j}$$
$$+ \underbrace{\frac{1}{r_i^2(\Delta\varphi)^2}}_{\gamma} u_{i,j+1} + \underbrace{\frac{1}{r_i^2(\Delta\varphi)^2}}_{\delta} u_{i,j-1}.$$

Clearly Γ, α, γ and δ are all positive, we have that

$$\beta = \frac{1}{(\Delta r)^2}\left(1 - \frac{1}{2i-1}\right) \geq 0,\ \text{since } i \geq 1.$$

We have that $\alpha + \beta + \gamma + \delta = \Gamma$ and thus u_{ij} is a convex combination of $u_{i\pm1,j\pm1}$ and the inequalities hold.

3c

Show that this implies that

$$\min_{\varphi\in[0,2\pi]} g(\varphi) \leq u_{ij} \leq \max_{\varphi\in[0,2\pi]} g(\varphi).$$

Possible solution: u_{ij} cannot have any local extrema hence the extrema must be on the boundary.

THE END