MAT 3360: First mandatory assignment, spring 2020

To be handed in by Thursday, February 13th., 14:30

You must hand in one (.pdf) file containing your answers as well as commented scripts which actually compile and work. In order to pass, you must have a score of at least 60% Each question (**a**), **b)** etc.) counts 10%.

You must also use "**Canvas**" to hand in the assignment.

Exercise 1. We study the boundary value problem

$$\begin{cases} u(x) - u''(x) = f(x), & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

Here f is a given function in $C([0,1])$. Observe that if u is a solution then

$$u(x) = \int_0^1 G(x,y)(f(y) - u(y))\, dy, \tag{2}$$

where G is defined as

$$G(x,y) = \begin{cases} y(1-x) & 0 \le y \le x, \\ x(1-y) & x \le y \le 1. \end{cases}$$

a) Show the converse, i.e., that if u satisfies (2), then u satisfies (1).

Define $\mathcal{H}$ as the mapping from $C([0,1])$ to $C_0([0,1])$ by

$$\mathcal{H}(v)(x) = \int_0^1 G(x,y)(f(y) - v(y))\, dy,$$

for any $v \in C([0,1])$.

b) Show that

$$\|\mathcal{H}(v) - \mathcal{H}(u)\|_\infty \le \frac{1}{8} \|u - v\|_\infty$$

for u and v in $C([0,1])$.

c) Define the sequence of functions $\{u^n\}_{n=0}^\infty$ in $C([0,1])$ by

$$\begin{cases} u^0(x) = 0, \\ u^{n+1}(x) = \mathcal{H}(u^n)(x), & n \ge 0. \end{cases}$$

Show that $u(x) = \lim_{n\to\infty} u^n(x)$ exists and that u solves (1).

d) Show that

$$\|u\|_\infty \le \frac{1}{7} \|f\|_\infty$$

Why does this imply that (1) has a unique solution if $f \in C([0,1])$?

We can also show that the solutions to (1) are unique if $\int_0^1 f^2\,dx < \infty$, this is a weaker condition on f. The next questions concern estimates available if f is square integrable.

e) Define

$$\|g\|_2 = \left(\int_0^1 g^2(y)\,dy\right)^{\frac{1}{2}}.$$

Show that $\|u\|_2^2 + 2\,\|u'\|_2^2 \leq \|f\|_2^2$. (Hint: It may be useful to observe that $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$.) Note that $\|\cdot\|_2$ is a norm, in particular the triangle inequality holds $\|g+h\|_2 \leq \|g\|_2 + \|h\|_2$. You don't have to show this.

f) Set $v = u'$, and show that

$$\|v'\|_2 \leq 2\,\|f\|_2\,.$$

In the next question the Cauchy-Schwartz inequality

$$\Big|\int_a^b g\,h\,dx\Big| \leq \Big(\int_a^b g^2\,dx\Big)^{\frac{1}{2}}\Big(\int_a^b h^2\,dx\Big)^{\frac{1}{2}}$$

will be useful.

g) Show that

$$|v(x) - v(y)| \leq 2\sqrt{|x-y|}\,\|f\|_2 \quad \text{for } x \text{ and } y \text{ in } [0,1].$$

h) Find the solution to (1) for $f(x) = (1+\pi^2)\sin(\pi x)$.

i) Use the approximation

$$u''(x) \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

to design a numerical scheme to solve (1) approximately.

j) Implement this scheme on a computer, and test it for the example from **h)**. Plot the approximate and exact solutions with $h = 1/11$.