Math 3360 Spring 2023 Oblig 1

Submission deadline

Thursday 2 March 2023, 14:30 through Canvas.

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). **The assignment must be submitted as a single PDF file.** Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all the necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. (Add the code to the single pdf.) You can use your programming language of choice.

There is only one attempt to pass the assignment and you must have a score of at least 60% to pass it.

Application for postponement

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail:`studieinfo@math.uio.no`) well before the deadline.

Both mandatory assignments in this course must be approved in the same semester before you are allowed to take the final examination.

Complete guidelines on compulsory assignments

For further details on the hand in of compulsory assignments, see:
`https://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html`

Exercises

Exercise 1.

Solve the following PDE using the method of characteristics:

a)

$$\begin{aligned} u_t(x,t) + x u_x(x,t) &= t - 1 \qquad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= \exp(-x^2) \qquad x \in \mathbb{R}. \end{aligned}$$

b)

$$\begin{aligned} u_t(x,t) + (x+t) u_x(x,t) &= 0 \qquad x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= \sin(x). \end{aligned}$$

Background for exercise 2 on big $\mathcal{O}$ notation (see TW Project 1.1 for more):

Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $\{z_n\}_{n=1}$ a sequence of positive real numbers. If there exists a constant $c > 0$ such that

$$|y_n| \leq c z_n \qquad \forall n \geq 1,$$

then $\{y_n\}$ is said to be of the order of $\{z_n\}$. This is compactly expressed by writing

$$y_n = \mathcal{O}(z_n).$$

Rate of convergence: When a sequence y_n converges towards $\bar{y}$ as n tends to infinity, we define its rate of convergence as the largest real value $\alpha > 0$ such that

$$y_n - \bar{y} = \mathcal{O}(n^{-\alpha}) \qquad \text{holds.}$$

For example, the sequence $y_n = n^{-3/2}$ converges towards 0 with rate $3/2$, while $y_n = \cos(\sqrt{1/n})$ converges towards 1 with rate $1/2$.

Let us also remark that

$$y_n - \bar{y} = \mathcal{O}(n^{-\alpha})$$

is equivalent to writing

$$y_n = \bar{y} + \mathcal{O}(n^{-\alpha}).$$

Exercise 2.

The following sequences converge towards 0 as n tends to infinity. Estimate their rate of convergence using, for example, Taylor expansions:

a) $y_n = \exp(1/n) - 1$

b) $y_n = \sin^2(n^{-1/3})$

Background for exercise 3:

For sequences of numerical approximations, it is often more natural to parameterize in a resolution parameter $h = 1/n$ than n, and to study the convergence as $h \downarrow 0$ rather than $n \to \infty$. A standard example is the following difference approximation of the derivative of a function:

$$y_h := \frac{f(x+h) - f(x)}{h} \approx f'(x).$$

Let y_h be a sequence that converges towards $\bar{y}$ as h tends to 0. Then the notation $y_h - \bar{y} = \mathcal{O}(h^\alpha)$, or equivalently $y_h = \bar{y} + \mathcal{O}(h^\alpha)$ means there exist a $c > 0$ such that

$$|y_h - \bar{y}| < ch^\alpha \qquad \text{for all sufficiently small } h > 0.$$

The rate of convergence is defined as the largest $\alpha > 0$ such that $y_h - \bar{y} = \mathcal{O}(h^\alpha)$.

Exercise 3.

When conducting numerical experiments, one often only has access to the approximation error $e_h := |y_h - \bar{y}|$ for some values of $h > 0$, and one seeks to find the convergence rate $\alpha > 0$ of the method, meaning the exponent such that the following approximately holds for all sufficiently small $h > 0$:

$$e_h \approx ch^\alpha .$$

a) Suppose that the above approximation indeed is an equality:

$$e_h = ch^\alpha, \tag{1}$$

and show that it then holds that

$$\alpha = \frac{\log(e_{h_1}/e_{h_2})}{\log(h_1/h_2)} \tag{2}$$

for any real values $h_1 \neq h_2$.

b) For a given function $f(x)$, we consider the following two approximations of $f'(0)$:

$$y_h = \frac{f(h) - f(0)}{h} \qquad \textbf{(forward difference)}$$

and

$$\hat{y}_h = \frac{f(h) - f(-h)}{2h} \qquad \textbf{(central difference)}$$

with the corresponding approximation error sequences $e_h = |y_h - f'(0)|$ and $\hat{e}_h = |\hat{y}_h - f'(0)|$.

For a sequence of values $h = 2^{-4}, 2^{-5}, \ldots, 2^{-10}$, use subsequent h-values and formula (2) to estimate the convegence rate for $y_h \to f'(0)$ and $\hat{y}_h \to f'(0)$ on a computer for the following two functions:

1. $f_1(x) = \exp(x/2)$
2. $f_2(x) = \sqrt{|x|}\sin(x)$ where we define $f_2'(0) = \lim_{x \to 0} f_2'(x) = 0$.

Interpret the results.

c) This is a short exercise on using loglog plots for estimating convergence rates of numerical methods.

As a visually more appealing alternative to the approach in Exercise 3 b), the convergence rate α can also be obtained from reading the slope of a curve in a so-called "loglog plot" (a plot that is log-scaled along both the x- and y-axis). For both of the functions considered in 3 b), plot the vector of h-values against the vector of e_h-values in a loglog plot and also plot h against $\hat{e}_h$ in the same loglog plot. The slope of these loglog-scaled curves are approximately given by

$$\frac{\log(e_{h_1}) - \log(e_{h_2})}{\log(h_1) - \log(h_2)}$$

and

$$\frac{\log(\hat{e}_{h_1}) - \log(\hat{e}_{h_2})}{\log(h_1) - \log(h_2)}$$

for two small real values $h_1 \neq h_2$. Show that the slopes obtained in these plots are close to the corresponding α-values obtained in 3 b) (introducing, for example, reference curves in your plot).

The example below illustrates how a loglog plot is made in Matlab:

```
h = 2.^(-(4:10));

%example sequences e_h and hatE_h
e_h =  h.^(1/2); hatE_h = h.^(5/4);
% loglog plot
loglog(h, e_h, h, hatE_h)
```

Exercise 4.

At each point $x \in (0,1)$ an elastic string subjected to a force

$$f(x;a) = -\frac{1}{\sqrt{2\pi}a} \exp\left(-\frac{(x-0.5)^2}{2a^2}\right),$$

where $a > 0$ is a force parameter that can be adjusted. The string is fixed at the endpoints and the displacement from its stationary state 0 at the point $x \in (0,1)$ is given by $u(x)$.

For a given $a > 0$, the string displacement is described by the differential equation

$$\begin{aligned} -u''(x) &= f(x;a) \qquad x \in (0,1) \\ u(0) &= u(1) = 0 \end{aligned} \tag{3}$$

a) Show that $u(x) \leq 0$ for all $x \in (0,1)$.

Hint: Use the Green's function solution representation and properties of the Green's function.

b) Show that for any $a > 0$, the maximial displacement of the solution of (3) satisfies the inequality:

$$\sup_{x \in [0,1]} |u(x)| \leq 1/4$$

Hint: Show first that $G_{\max} := \max_{(x,y)\in[0,1]^2} G(x,y) = 1/4$, and use that $-f(x;a)$ is the probability density function of a normal distribution.

c) To solve equation (3) numerically we introduce $h = 1/(n+1)$, the uniform mesh $x_i = ih$ for $i = 0, 1, \ldots, n+1$, the difference approximation

$$-\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} = f(x_j; a), \qquad j = 1, 2, \ldots, n \tag{4}$$

and the boundary conditions $v_0 = v_{n+1} = 0$.

Write (4) as a system of linear equations $Av = b$ and solve the differential equation numerically using $n = 100$ for the parameter value $a = 1$. Plot the numerical solution and compute the maximal displacement:

$$\|v\|_{h,\infty} := \max_{j=0,1,\ldots,n+1} |v_j|.$$

d) Find a force $f(\cdot; a)$ such that the maximal displacement equals $1/5$. That is, approximate a value of $a > 0$ such that the maximal displacement of the numerical solution of the differential equation (3) equals $1/5$.

Comment: Due to the form of f, there is no closed form solution for this problem. Any approach leading to a good approximation of a is fine. You may for instance first use the quite coarse resolution $n = 100$, as considered for the numerical solutions in the previous exercise (but note that the answer may not be accurate for tiny values of a when using that resolution).

Exercise 5.

We look at the Poisson equation

$$\begin{aligned} -u''(x) &= f(x) \qquad x \in (0,1) \\ u(0) &= u(1) = 0 \end{aligned} \tag{5}$$

where we assume that $f \in C^4([0,1])$. The purpose of this exercise is to study the convergence rate of two numerical methods. Let the resolution parameters h and n and the grid x_j be defined as in Exercise 4 c) and consider:

- the standard method

$$-\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} = f(x_j), \qquad j = 1, 2, \ldots, n \tag{6}$$

  with boundary conditions $v_0 = v_{n+1} = 0$,

- and the higher order method

$$-\frac{\bar{v}_{j+1} - 2\bar{v}_j + \bar{v}_{j-1}}{h^2} = f(x_j) + \frac{h^2}{12} f''(x_j), \qquad j = 1, 2, \ldots, n \tag{7}$$

  again with boundary conditions $\bar{v}_0 = \bar{v}_{n+1} = 0$.

Theorem 2.2 in TW shows that for the standard numerical method $\|v - u\|_{h,\infty} = \mathcal{O}(h^2)$, and in exercise 5 a) and b) below we will derive the convergence rate for the higher order method.

a) The truncation error for the higher order method is given by

$$\bar{\tau}_h(x_j) = -\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} - \left(f(x_j) + \frac{h^2}{12} f''(x_j)\right) \quad j = 1, \ldots, n\,.$$

Use Taylor expansions to show that

$$|\bar{\tau}_h(x_j)| \leq \frac{\|f^{(4)}\|_\infty}{360} h^4 \quad \text{for all} \quad j = 1, \ldots, n \tag{8}$$

where $f^{(4)}(x) = \frac{d^4}{dx^4} f(x)$.

b) Use (8) to show that

$$\|\bar{v} - u\|_{h,\infty} \leq C \|f^{(4)}\|_\infty h^4,$$

and determine the value of the constant C.

c) Equation (5) with the force term $f(x) = e^x(1 + x)$ has the exact solution $u(x) = (1 - x)(e^x - 1)$.

Solve this equation using both of the numerical methods considered above on a sequence of resolution values $n = 2^2, 2^3, \ldots, 2^8$ and verify numerically that

$$\|v - u\|_{h,\infty} = \mathcal{O}(h^2) \quad \text{and} \quad \|\bar{v} - u\|_{h,\infty} = \mathcal{O}(h^4).$$