Math 3360 Spring 2023 Oblig 2

Submission deadline

Thursday 4 May 2023, 14:30 through Canvas.

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). **The assignment must be submitted as a single PDF file.** Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all the necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. (Add the code to the single pdf.) You can use your programming language of choice.

There is only one attempt to pass the assignment and you must have a score of at least 60% to pass it.

Application for postponement

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail:`studieinfo@math.uio.no`) well before the deadline.

Both mandatory assignments in this course must be approved in the same semester before you are allowed to take the final examination.

Complete guidelines on compulsory assignments

For further details on the hand in of compulsory assignments, see:
`https://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html`

Exercises

Exercise 1.

In this exercise, we will apply the method of characteristics to a boundary value problem.

Compute the solution of the transport equation

$$\begin{aligned} u_t + u_x &= 0 \qquad x \in (0,1), \quad t > 0 \\ u(x,0) &= f(x) \qquad x \in (0,1) \end{aligned} \tag{1}$$

with the boundary condition

$$u(0,t) = g(t) \qquad t \geq 0, \tag{2}$$

where we assume that $f \in C^1([0,1))$ and $g \in C^1([0,\infty))$ and

$$f(0) = g(0), \qquad f'(0) = -g'(0). \tag{3}$$

And explain why (3) is a necessary condition to ensure that the solution u belongs to $C^1([0,1) \times [0,\infty))$.

Hint: If applying the method of characteristics $(s, x(s))$ and the mapping $x_0(x,t)$ is such that $x_0(x,t) < 0$ for a point $(x,t) \in (0,1) \times [0,\infty)$, then the initial condition $f(x)$ can not be used to determine the value of $u(x,t)$, but one can instead make use of the "left-side mapping" $t_0(x,t) = \{s \in [0,t] | x(s) = 0\}$.

Exercise 2.

Show that if $u(x,y)$ is a smooth solution of the PDE

$$(1+x^2)u_{xx}(x,y) + (1+x^2)u_{yy}(x,y) + 4xu_x(x,y) + 2u(x,y) = 1 \qquad (x,y) \in \Omega = (0,1)^2$$

with boundary condition $u(x,y) = 0$ on $\partial\Omega = [0,1]^2 \setminus (0,1)^2$, then it holds that

$$\max_{(x,y)\in[0,1]^2} |u(x,y)| \leq \frac{1}{8}.$$

Exercise 3.

We consider the following heat equation in 2-dimensional physical space:

$$\left.\begin{aligned} u_t(x,y,t) &= u_{xx}(x,y,t) + u_{yy}(x,y,t) \qquad && (x,y,t) \in (0,1)^2 \times (0,T] \\ u(0,y,t) = u(1,y,t) &= u(x,0,t) = u(x,1,t) = 0 \qquad && (x,y,t) \in [0,1]^2 \times [0,T] \end{aligned}\right\} \tag{4}$$

and with initial condition

$$u(x,y,0) = f(x,y) \qquad (x,y) \in (0,1)^2, \tag{5}$$

where $f \in C^2([0,1]^2)$. Let the function domain be denoted $R = [0,1]^2 \times [0,T]$, and let the "lower" boundary of the domain be denoted $B = R \setminus ((0,1)^2 \times (0,T])$.

a) Show that (4)-(5) has at most one solution that belongs to $u \in C^{2,1}((0,1)^2 \times (0,T]) \cap C(R)$.

b) Determine and solve the ODE for $T_{k_1,k_2}(t)$ with initial condition $T_{k_1,k_2}(0) = 1$ such that

$$u_{k_1,k_2}(x, y, t) = T_{k_1,k_2}(t) \sin(k_1 \pi x) \sin(k_2 \pi y) \quad k_1, k_2 \in \mathbb{N}$$

is a family of particular solutions of (4)

c) Compute the unique solution to (4) when $f(x, y) = \sin(3\pi x) \sin(4\pi y)$.

d) Using the step sizes $h = 1/(n+1)$ and $\Delta t = T/(N+1)$, and the set of mesh points

$$(x_j, y_k, t_m) = (jh, kh, m\Delta t) \qquad 0 \le j, k \le n+1, \quad 0 \le m \le N+1, \tag{6}$$

a numerical solution $v_{j,k}^m = v(x_j, y_k, t_m)$ is obtained by the explicit scheme

$$\frac{v_{j,k}^{m+1} - v_{j,k}^m}{\Delta t} = \frac{v_{j-1,k}^m + v_{j+1,k}^m + v_{j,k-1}^m + v_{j,k+1}^m - 4v_{j,k}^m}{h^2} \tag{7}$$

for $1 \le j, k \le n$ and $0 \le k \le N$, and with initial condition

$$v_{j,k}^0 = f(x_j, y_k) \qquad 1 \le j, k \le n$$

and boundary condition

$$v_{0,k}^m = v_{n+1,k}^m = v_{j,0}^m = v_{j,n+1}^m = 0 \qquad 0 \le j, k \le n+1 \quad 0 \le m \le N+1.$$

The exercise: Let the set of all mesh points in (6) be denoted R_Δ and let the set of mesh points on the "lower" boundary be denoted by B_Δ. That is,

$$B_\Delta = \{(x_j, y_k, t_m) \in R_\Delta \mid (x_j, y_k, t_m) \in B\}.$$

Show that if

$$r := \frac{\Delta t}{h^2} \le 1/4\,,$$

then it holds that

$$\max_{(j,k,m) \in R_\Delta} |v_{j,k}^m| \le \max_{(j,k) \in \{0,1,\ldots,n+1\}^2} |f(x_j, y_k)|.$$

Why does this imply that the numerical solution is unique?

e) Solve (4) with the initial condition $f(x, y) = 23x(1-x)y(1-y)$ numerically using $h = 1/100$ and $\Delta t = h^2/4$, and find the smallest time $t_m > 0$ such that

$$\max_{(j,k) \in \{0,1,\ldots,n+1\}^2} v(x_j, y_k, t_m) < \frac{1}{\pi^2}.$$