Exercises: 3.8, 3.10, 3.11, 3.12, 3.14, 3.15

3.8

$$u_t = u_{xx} - u \qquad (x,t) \in (0,1) \times (0,\infty)$$
$$u(0,t) = u(1,t) = 0$$

Guess $u_k(x,t) = T_k(t)\, X_k(x)$

PDE gives $T_k'\, X_k = (X_k'' - X_k)\, T_k$

$$\Rightarrow \quad \frac{T_k'}{T_k} = \frac{X_k'' - X_k}{X_k} = -\lambda_k$$

for some $\lambda_k \in \mathbb{R}$.

Eigenval problem

$$-X_k'' = (\lambda_k - 1) X_k =: \bar{\lambda}_k X_k$$
$$X_k(0) = X_k(1) = 0$$

$$\Rightarrow \quad \bar{\lambda}_k = (k\pi)^2, \quad X_k = \sin(k\pi x)$$

$$\& \quad \lambda_k = (k\pi)^2 + 1$$

Eq for T_k: $\quad T_k' = -\lambda_k T_k$

$$\Rightarrow T_k(t) = e^{-\lambda_k t} \underbrace{T_k(0)}_{\text{set} = 1} = e^{-((k\pi)^2 + 1)t}.$$

Family of sols:

$$u_k(x,t) = e^{-((k\pi)^2+1)t} \sin(k\pi x) \qquad k \in \mathbb{N}.$$

3.10 Formal sol to

$$u_t = u_{xx} \qquad x \in (0,\ell), \quad t>0$$

$$u(0,t) = u(\ell,t) = 0$$

$$u(x,0) = f(x)$$

Define $V(x,t) = u(\ell x, t) \quad x \in (0,1), t \geq 0.$

Then $V_t = u_t, \quad V_{xx} = \ell^2 u_{xx}$

$$\Rightarrow \quad V_t = \ell^{-2} V_{xx}$$

$$V(0,t) = V(1,t) = 0$$

$$V(x,0) = u(\ell x, 0) = f(\ell x)$$

Particular solutions for V:

$$V_k(x,t) = X_k(x)\, T_k(t)$$

$$\Rightarrow \frac{T_k'(t)}{T_k(t)} = \ell^{-2} \frac{X_k''(x)}{X_k(x)} = -\lambda_k$$

Eigenval problem:

$$-X_k'' = \ell^2 \lambda_k X_k =: \overline{\lambda_k} X_k \qquad x \in (0,1)$$

$$X_k(0) = X_k(1) = 0$$

Solution $\overline{\lambda_k} = (k\pi)^2$, $\lambda_k = \left(\frac{k\pi}{\ell}\right)^2$

& $X_k = \sin(k\pi x)$.

$$T_k' = -\lambda_k T_k \Rightarrow T_k(t) = e^{-\left(\frac{k\pi}{\ell}\right)^2 t} \quad \sin(k\pi x)$$

using $T_k(0) = 1$.

Formal solution v:

$$v(x,t) = \sum_{k=1}^{\infty} c_k \, e^{-\left(\frac{k\pi}{\ell}\right)^2 t} \sin(k\pi x)$$

$$c_k = 2 \int_0^1 f(\ell x) \sin(k\pi x) \, dx$$

$$\left(= \frac{2}{\ell} \int_0^\ell f(x) \sin\left(\frac{k\pi x}{\ell}\right) dx \right)$$

Formal solution u

$$u(x,t) = v\left(\frac{x}{\ell}, t\right)$$

$$= \sum_{k=1}^{\infty} c_k e^{-\left(\frac{k\pi}{\ell}\right)^2 t} \sin\left(\frac{k\pi x}{\ell}\right)$$

3.11 Formal sol to

$$u_t = u_{xx} \qquad x \in (0,1) \quad t > 0$$
$$u(0,t) = a \quad u(1,t) = b$$
$$u(x,0) = f(x)$$

Let $V(x,t) = u(x,t) - (a + (b-a)x)$

$\Rightarrow$
$$V_t = V_{xx}$$
$$V(0,t) = 0, \quad V(1,t) = 0$$
$$V(x,0) = f(x) - (a + (b-a)x)$$

Formal sol V:
$$V(x,t) = \sum_{k=1}^{\infty} C_k e^{-(k\pi)^2 t} \sin(k\pi x)$$
$$C_k = 2\int_0^1 \left(f(x) - (a + (b-a)x)\right) \sin(k\pi x)\, dx$$

and formal sol u:
$$u(x,t) = V(x,t) + (a + (b-a)x)$$

Ex 3.12 Formal sol to

$$u_t = u_{xx} + 2x$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = f(x)$$

Let $V(x,t) = u(x,t) + \dfrac{x^3 - x}{3}$

$\Rightarrow$ $V_t = u_t$ & $V_{xx} = u_{xx} + 2x$

$$\Rightarrow \quad V_t = V_{xx}$$
$$V(0,t) = V(1,t) = 0$$
$$V(x,0) = f(x) + \frac{x^3-x}{3} = \tilde{f}(x)$$

Standard next steps:

Formal sol V:

$$V(x,t) = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} \sin(k\pi x)$$

$$c_k = 2\int_0^1 \tilde{f}(x) \sin(k\pi x)\, dx.$$

Formal sol u:

$$u(x,t) = V(x,t) - \frac{x^3-x}{3}.$$

Exercise 3.14 (a)

$$\int_0^1 \cos(k\pi x)\cos(m\pi x)\,dx = \begin{cases} 0 & k \neq m \\ \frac{1}{2} & k = m \geq 1 \\ 1 & k = m = 0 \end{cases}$$

Assume $k \leq m$. • If $k = 0$ then

$$\int_0^1 \cos(m\pi x)\,dx = \begin{cases} 1 & m = 0 \\ 0 & \text{otherwise.} \end{cases}$$

• $k \geq 1$ and $m = k$, then

$$\int_0^1 \cos^2(k\pi x)\,dx = \int_0^1 \frac{1 - \cos(2k\pi x)}{2}\,dx$$

$$= \frac{1}{2}$$

• $k \geq 1$ & $m \neq k$, then

$$\cos(k\pi x)\cos(m\pi x)$$

$$= \frac{\cos((m+k)\pi x) + \cos((m-k)\pi x)}{2} \quad (*)$$

$$\Rightarrow \int_0^1 \cos(k\pi x)\cos(m\pi x)\,dx$$

$$= \frac{1}{2}\int_0^1 \cos((m+k)\pi x) + \cos((m-k)\pi x)\,dx = 0.$$

(b) Hints for exercise:

Let now $\langle f, g \rangle := \int_{-1}^{1} f(x) g(x)\, dx$

$$\langle \sin(k\pi x), \sin(m\pi x) \rangle = \begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases}$$

show using

$$\sin(k\pi x)\sin(m\pi x) = \frac{\cos((k-m)\pi x) - \cos((k+m)\pi x)}{2}$$

$$\langle \cos(k\pi x), \cos(m\pi x) \rangle = \begin{cases} 0 & k \neq m \\ 1 & k = m \geq 1 \\ 2 & k = m = 0 \end{cases}$$

Use $(*)$ from (a).

$$\langle \sin(k\pi x), \cos(m\pi x) \rangle = 0$$

Use

$$\sin(k\pi x)\cos(m\pi x) = \frac{\sin((k+m)\pi x) + \sin((k-m)\pi x)}{2}$$

3.15 (a) $-X'' = \lambda X \qquad x \in (-1,1)$

Periodic BC: $X(-1) = X(1), \quad X'(-1) = X'(1)$

$$\langle -X'', X\rangle = -\int_{-1}^{1} X''(x) X(x)\,dx = \left[-X' X\right]_{-1}^{1} + \int_{-1}^{1} (X')^2\,dx$$

$$\Rightarrow \quad \lambda = \frac{\langle X, X\rangle}{\langle X', X'\rangle} \geq 0$$

writing $\beta = \sqrt{\lambda}$ yields problem

$$X'' = -\beta X$$

General sol $X(x) = a\cos(\beta x) + b\sin(\beta x)$

and $\quad X'(x) = \beta\left(b\cos(\beta x) - a\sin(\beta x)\right)$

$X(-1) = X(1) \Rightarrow b\sin(\beta) = 0$

$X'(-1) = X'(1) \Rightarrow \beta a \sin(\beta) = 0$

$\Rightarrow \beta_k = k\pi, \quad \lambda_k = (k\pi)^2$

with eigenfunctions

$X_k(x) = \cos(k\pi x), \quad X_k(x) = \sin(k\pi x).$

(b) By orthogonality in 3.14 (b),

$$\langle f, \sin(k\pi x)\rangle = \frac{a_0}{2}\underbrace{\langle 1, \sin(k\pi x)\rangle}_{=0}$$

$$+ \sum_{k=1}^{\infty} a_k \underbrace{\langle \cos(k\pi x), \sin(k\pi x)\rangle}_{=0}$$

$$+ \sum_{k=1}^{\infty} b_k \underbrace{\langle \sin(k\pi x), \sin(m\pi x)\rangle}_{=\begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases}}$$

$$= b_k$$

Derive $a_k = \langle f, \cos(k\pi x)\rangle$ similarly.

(c) Guess $u_k(x,t) = X_k T_k$ yields eigenval problem

$$-X_k'' = \lambda_k X_k, \quad X_k(-1) = X_k(1)$$
$$X_k'(-1) = X_k'(1)$$

Part (a) $\Rightarrow$ $\lambda_k = (k\pi)^2$

$X_k(x) = \cos(k\pi x)$, $X_k(x) = \sin(k\pi x)$.

ODE for T_k:

$$T_k' = -\lambda_k T_k \quad , \quad T_k(0) = 1$$

$$\Rightarrow T_k(t) = e^{-(k\pi)^2 t}.$$

Particular solutions

$$u_k(x,t) = e^{-(k\pi)^2 t} \sin(k\pi x) \qquad k \geq 1$$

and

$$v_k(x,t) = e^{-(k\pi)^2 t} \cos(k\pi x) \qquad k \geq 0.$$

Formal sol

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k v_k(x,t) + \sum_{k=1}^{\infty} b_k u_k(x,t)$$

where a_k, b_k are given in (b).