MAT 3360: Exam, spring 2020

Deadline: Friday June 12th., 14:30

You must hand in one (and only one) (.pdf) file. Passing requires a score of at least 40% where each question (**a**), **b**) etc.) counts 10%.

Question 1. Set

$$f(x) = x(1 - x^2).$$

a) Find the Fourier sine series of f in the interval $[0, 1]$.

b) Denote the sum of the Fourier series by $S(x)$. Sketch the graph of S for $x \in [-2, 2]$.

c) Show that

$$\pi^6 = 945 \sum_{k=1}^{\infty} \frac{1}{k^6}.$$

Question 2. Consider the initial value problem

$$\begin{cases} u_t + \cos^2(x) u_x = 0, & t > 0, \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ u(x, 0) = f(x), \end{cases} \tag{1}$$

where f is a known continuously differentiable function. We assume that u is continuously differentiable (both in x and t) function.

a) Set

$$E(t) = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (u(x, t))^2 \, dx.$$

Show that $E'(t) \leq 2E(t)$ and therefore that $E(t) \leq E(0)e^{2t}$.

b) Explain why **a)** implies that (1) has a most one continuously differentiable solution.

c) Use the method of characteristics to find a formula for the solution of (1) for a given function f.

Question 3. Consider the boundary value problem

$$\begin{cases} \Delta u = 0, & 1 < r < 2, \\ u(1, \theta) = 0, & u(2, \theta) = \sin(\theta). \end{cases} \tag{2}$$

Here (r, θ) denote polar coordinates and the Laplace operator Δ is given by

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

You shall not show this.

a) Use separation of variable to find the solution $u(r, \theta)$.

b) Find the solution to the boundary value problem

$$\begin{cases} \Delta v = 0, \quad 1 < r < 2, \\ v(1, \theta) = \cos(2\theta), \quad v(2, \theta) = 0. \end{cases} \tag{3}$$

c) Solve the boundary value problem

$$\begin{cases} \Delta w = 0, \quad 1 < r < 2, \\ w(1, \theta) = \cos(2\theta), \quad w(2, \theta) = \sin(\theta). \end{cases} \tag{4}$$

Question 4. We shall consider the Crank-Nicholson scheme for the heat equation

$$\begin{cases} u_t = u_{xx}, \ t > 0, \ x \in (0, 1), \\ u(0, t) = u(1, t) = 0, \ t > 0, \\ u(x, 0) = f(x). \end{cases}$$

This scheme can be re-written

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} - \frac{1}{\Delta x^2} D^+ D^- v_j^{m+1/2} = 0, \quad j = 1, \dots, n,$$

with $v_0^m = v_{n+1}^m = 0$, where $\Delta x = 1/(n + 1)$ and

$$v_j^{m+1/2} := \frac{1}{2} \left(v_j^{m+1} + v_j^m \right),$$

and the differences D^+ and D^- are defined as

$$D^+ a_j = a_{j+1} - a_j, \quad D^- a_j = a_j - a_{j-1}.$$

Define the discrete energy E^m by

$$E^m = \frac{1}{2} \Delta x \sum_{j=1}^{n} \left(v_j^m \right)^2.$$

Show that $E^{m+1} \leq E^m$ for $m \geq 0$. (**Hint**: Multiply scheme with $v_j^{m+1/2}$ and sum over j.)