UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT3360 — Introduction to partial differential equations

Day of examination: Friday, June 11, 2021

Examination hours: 09:00 – 13:00

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weigth 15%)

Consider the PDE

$$\begin{cases} u_t + (1+x^2)u_x = 0, \quad t>0, \quad x \in \mathbb{R}, \\ u(x,0) = \frac{1}{1+x^2}. \end{cases}$$

Find a solution to this initial value problem.

Problem 2 (weigth 25%)

Consider the function $f : [-1,1] \mapsto \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{\sin(\pi x)}{x} & x \neq 0, \\ \pi & x = 0. \end{cases}$$

We have that the full Fourier series of f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x).$$

2a

Explain why the Fourier series converges uniformly to f for $x \in [-1,1]$, and converges uniformly to a function g for $x \in \mathbb{R}$. Draw the graph of g for $x \in [-3,3]$.

2b

Show that $b_k = 0$ and that

$$a_k = \int_{k-1}^{k+1} \frac{\sin(\pi x)}{x}\, dx, \quad k = 0,1,2,3,\ldots$$

(Continued on page 2.)

2c

Use the Fourier series of f to calculate the improper integral

$$\int_0^\infty \frac{\sin(\pi x)}{x}\, dx.$$

Problem 3 (weigth 30%)

Let $Q(x)$ be a function in $C_0^2((0,1))$.
For $k = 1, 2, 3, \ldots$ define $X_k(x) = \sin(k\pi x)$.

3a

Define

$$u_N(x,t) = 2\int_0^t \int_0^1 \sum_{k=1}^{N} Q(y)X_k(x)X_k(y)e^{-(k\pi)^2(t-s)}\, dyds.$$

Show that u_N is a solution of the boundary value problem

$$\begin{cases} \frac{\partial}{\partial t}u_N - \frac{\partial^2}{\partial x^2}u_N = Q_N & t \in (0,T],\ x \in (0,1), \\ u_N(0,t) = u_N(1,t) = 0 & t > 0, \\ u_N(x,0) = 0, \end{cases}$$

where

$$Q_N(x) = 2\sum_{k=1}^{N} X_k(x) \int_0^1 X_k(y)Q(y)\, dy.$$

3b

Show that $Q_N \to Q$ uniformly in $[0,1]$.

3c

Assume that there exists a smooth solution u to the problem

$$\begin{cases} \frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u = Q & t \in (0,T],\ x \in (0,1), \\ u(0,t) = u(1,t) = 0 & t > 0, \\ u(x,0) = 0, \end{cases}$$

Set $E(t) = \|u(\cdot,t)\|$, where $\|\cdot\|$ denotes the mean square norm.
Show that

$$E(t) \leq t\,\|Q\|$$

3d

Show that u_N converges in the mean square norm to u as $N \to \infty$.

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Problem 4 (weigth 30%)

Consider the transport equation in the periodic setting

$$\begin{cases} u_t + u_x = 0, & t > 0,\ x \in [0,1], \\ u(0,t) = u(1,t) \\ u(x,0) = f(x), \end{cases} \tag{1}$$

where f is a given smooth periodic function with period 1.
Consider also the difference scheme

$$L_{\Delta x} v_j^m := \frac{v_j^{m+1} - \frac{1}{2}(v_{j+1}^m + v_{j-1}^m)}{\Delta t} + \frac{v_{j+1}^m - v_{j-1}^m}{2\Delta x} = 0,\ \ m \geq 0,\ j = 0,1,\dots,N,$$

and $v_{-1}^m = v_N^m$, $v_{N+1}^m = v_0^m$. The initial values are given by

$$v_j^0 = f(x_j).$$

Here Δt is a small positive number, $\Delta x = 1/(N+1)$ and $x_j = j\Delta x$. We also define $t^m = m\Delta t$. The scheme is explicit since we can solve for v_j^{m+1},

$$v_j^{m+1} = \frac{1}{2}\,(1-r)\,v_{j+1}^m + \frac{1}{2}\,(1+r)\,v_{j-1}^m,$$

with $r = \Delta t/\Delta x$.

4a

Find a condition on r which guarantees that

$$\min_j v_j^m \leq v_j^{m+1} \leq \max_j v_j^m$$

for $m \geq 0$. Assume from now on that r satisfies this condition.

4b

Assume that w_j^m solves

$$L_{\Delta x} w_j^m = g_j^m$$

for $m \geq 0$ and $j = 0,\dots,N$ with periodic boundary conditions $w_{-1}^m = w_N^m$, $w_{N+1}^m = w_0^m$. Here g_j^m is a given grid function. We assume that $w_j^0 = 0$ for all j. Show that

$$\max_{j=0,\dots,N} \left|w_j^m\right| \leq m\Delta t \max_{\substack{j=0,\dots,N \\ k=0,\dots,m-1}} \left|g_j^k\right|.$$

4c

Let u be a smooth solution of (1), show that

$$L_{\Delta x} u(x_j, t^m) = \mathcal{O}(\Delta x),$$

and use this to obtain a bound of the error

$$\max_{j=0,\dots,N} \left|v_j^m - u(x_j, t^m)\right|.$$

THE END