UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT3360 — Introduction to partial differential equations

Day of examination: Friday, June 11, 2021

Examination hours: 09:00 – 13:00

This problem set consists of 7 pages.

Appendices: None.

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weigth 15%)

Consider the PDE

$$\begin{cases} u_t + (1+x^2)u_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = \frac{1}{1+x^2}. \end{cases}$$

Find a solution to this initial value problem.

Løsningsforslag: The characteristic equation is

$$x' = (1+x^2), \quad x(0) = x_0,$$

with solution

$$x_0 = \tan\left(\arctan(x) - t\right).$$

Hence a solution of the PDE is

$$u(x,t) = \frac{1}{1 + \left(\tan\left(\arctan(x) - t\right)\right)^2}.$$

Problem 2 (weigth 25%)

Consider the function $f : [-1,1] \mapsto \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{\sin(\pi x)}{x} & x \neq 0, \\ \pi & x = 0. \end{cases}$$

We have that the full Fourier series of f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x).$$

(Continued on page 2.)

2a

Explain why the Fourier series converges uniformly to f for $x \in [-1,1]$, and converges uniformly to a function g for $x \in \mathbb{R}$. Draw the graph of g for $x \in [-3,3]$.

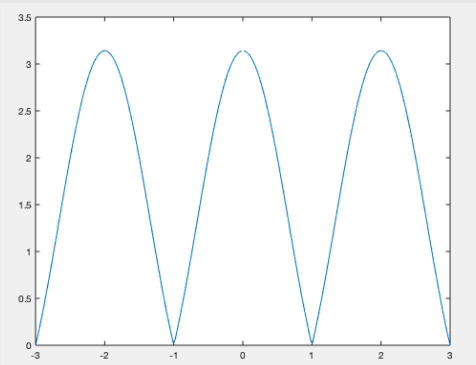
Løsningsforslag: We have that f is continuous on $[-1,1]$, since

$$\lim_{x\to 0} f(x) = \pi.$$

f' is continuous on $[-1,1]$ since

$$f'(0) = \lim_{h=0} \frac{f(h)-\pi}{h} = \lim_{h\to 0} \frac{\sin(\pi h) - \pi h}{h^2} = -\frac{\pi^2}{2} = \lim_{x\to 0} f'(x).$$

Hence the Fourier series for f' converges pointwise (to its periodic extension). Then the Fourier series for f will converge uniformly to the periodic extension of f.



2b

Show that $b_k = 0$ and that

$$a_k = \int_{k-1}^{k+1} \frac{\sin(\pi x)}{x}\, dx, \quad k = 0,1,2,3,\ldots$$

Løsningsforslag: $b_k = 0$ since f is even. Then

$$\begin{aligned} a_k &= 2\int_0^1 \frac{\sin(\pi x)\cos(k\pi x)}{x}\, dx \\ &= \int_0^1 \frac{\sin((k+1)\pi x) - \sin((k-1)\pi x)}{x}\, dx \\ &= \int_0^{k+1} \frac{\sin(\pi y)}{y}\, dy - \int_0^{k-1} \frac{\sin(\pi y)}{y}\, dy \\ &= \int_{k-1}^{k+1} \frac{\sin(\pi y)}{y}\, dy. \end{aligned}$$

(Continued on page 3.)

2c

Use the Fourier series of f to calculate the improper integral

$$\int_0^\infty \frac{\sin(\pi x)}{x}\,dx.$$

Løsningsforslag: We know that

$$\begin{aligned}
\pi = f(0) &= \lim_{N\to\infty} \frac{a_0}{2} + \sum_{k=1}^{N} a_k \\
&= \lim_{N\to\infty} \Big(\int_0^1 \frac{\sin(\pi x)}{x}\,dx + \sum_{k=1}^{N} \int_{k-1}^{k+1} \frac{\sin(\pi x)}{x}\,dx \Big) \\
&= \int_0^1 \frac{\sin(\pi x)}{x}\,dx + \lim_{N\to\infty} \sum_{k=0}^{N-1} \int_k^{k+1} \frac{\sin(\pi x)}{x}\,dx + \sum_{k=1}^{N} \int_k^{k+1} \frac{\sin(\pi x)}{x}\,dx \\
&= \lim_{N\to\infty} \int_0^N \frac{\sin(\pi x)}{x}\,dx + \lim_{N\to\infty} \int_0^{N+1} \frac{\sin(\pi x)}{x}\,dx \\
&= 2\int_0^\infty \frac{\sin(\pi x)}{x}\,dx.
\end{aligned}$$

Therefore $\int_0^\infty \frac{\sin(\pi x)}{x}\,dx = \pi/2$.

Problem 3 (weigth 30%)

Let $Q(x)$ be a function in $C_0^2((0,1))$.
For $k = 1, 2, 3, \ldots$ define $X_k(x) = \sin(k\pi x)$.

3a

Define

$$u_N(x,t) = 2\int_0^t \int_0^1 \sum_{k=1}^{N} Q(y) X_k(x) X_k(y) e^{-(k\pi)^2(t-s)}\,dyds.$$

Show that u_N is a solution of the boundary value problem

$$\begin{cases} \frac{\partial}{\partial t} u_N - \frac{\partial^2}{\partial x^2} u_N = Q_N & t \in (0,T],\ x \in (0,1), \\ u_N(0,t) = u_N(1,t) = 0 & t > 0, \\ u_N(x,0) = 0, \end{cases}$$

where

$$Q_N(x) = 2\sum_{k=1}^{N} X_k(x) \int_0^1 X_k(y) Q(y)\,dy.$$

(Continued on page 4.)

Løsningsforslag: We calculate

$$\frac{\partial}{\partial t}u_N = 2\int_0^1 \sum_{k=1}^N Q(y)X_k(x)X_k(y)\,dyds - 2\int_0^t\int_0^1 \sum_{k=1}^N Q(y)X_k(x)X_k(y)(k\pi)^2 e^{-(k\pi)^2(t-s)}\,dyds$$

and

$$\frac{\partial^2}{\partial x^2}u_N = -2\int_0^t\int_0^1 \sum_{k=1}^N Q(y)(k\pi)^2 X_k(x)X_k(y)e^{-(k\pi)^2(t-s)}\,dyds.$$

Hence u_N satisfies the differential equation. It is easy to see that the initial and boundary conditions are satisfied.

3b

Show that $Q_N \to Q$ uniformly in $[0,1]$.

Løsningsforslag: We recognise Q_N as the partial sum of the Fourier expansion of $Q(t)$

$$Q_N = \sum_{k=1}^N \frac{\langle Q, X_k\rangle}{\|X_k\|^2} X_k.$$

Since $Q \in C_0^2((0,1))$ we know that its Fourier series converges uniformly to $Q(x,t)$.

3c

Assume that there exists a smooth solution u to the problem

$$\begin{cases} \frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u = Q & t \in (0,T],\ x \in (0,1), \\ u(0,t) = u(1,t) = 0 & t > 0, \\ u(x,0) = 0, \end{cases}$$

Set $E(t) = \|u(\cdot,t)\|$, where $\|\cdot\|$ denotes the mean square norm.
Show that

$$E(t) \leq t\,\|Q\|$$

Løsningsforslag: We multiply with u and integrate over x to find

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 + \|u_x(\cdot,t)\|^2 = \langle u(\cdot,t), Q\rangle \leq \|u(\cdot,t)\|\,\|Q\|.$$

Hence

$$\frac{d}{dt}\|u(\cdot,t)\| \leq \|Q\|.$$

(Continued on page 5.)

3d

Show that u_N converges in the mean square norm to u as $N \to \infty$.

Løsningsforslag: We get that

$$\frac{\partial}{\partial t}(u - u_N) - \frac{\partial^2}{\partial x^2}(u - u_N) = (Q - Q_N).$$

Multiply this with $(u - u_N)$ and integrate to get

$$\frac{d}{dt}\|u(\cdot,t) - u_N(\cdot,T)\| \le \|Q - Q_N\|_\infty t.$$

Hence

$$\|u(\cdot,t) - u_N(\cdot,t)\| \le \frac{t^2}{2}\|Q - Q_N\|_\infty \to 0,$$

as $N \to \infty$.

Problem 4 (weigth 30%)

Consider the transport equation in the periodic setting

$$\begin{cases} u_t + u_x = 0, & t > 0,\ x \in [0,1], \\ u(0,t) = u(1,t) \\ u(x,0) = f(x), \end{cases} \tag{1}$$

where f is a given smooth periodic function with period 1.
Consider also the difference scheme

$$L_{\Delta x} v_j^m := \frac{v_j^{m+1} - \frac{1}{2}(v_{j+1}^m + v_{j-1}^m)}{\Delta t} + \frac{v_{j+1}^m - v_{j-1}^m}{2\Delta x} = 0,\ \ m \ge 0,\ j = 0, 1, \dots, N,$$

and $v_{-1}^m = v_N^m$, $v_{N+1}^m = v_0^m$. The initial values are given by

$$v_j^0 = f(x_j).$$

Here Δt is a small positive number, $\Delta x = 1/(N+1)$ and $x_j = j\Delta x$. We also define $t^m = m\Delta t$. The scheme is explicit since we can solve for v_j^{m+1},

$$v_j^{m+1} = \frac{1}{2}(1 - r)\, v_{j+1}^m + \frac{1}{2}(1 + r)\, v_{j-1}^m,$$

with $r = \Delta t/\Delta x$.

4a

Find a condition on r which guarantees that

$$\min_j v_j^m \le v_j^{m+1} \le \max_j v_j^m$$

for $m \ge 0$.

(Continued on page 6.)

Løsningsforslag: If $0 < r \le 1$ then v_j^{m+1} is a convex combination of $v_{j\pm 1}^m$ and therefore

$$\min_j v_j^m \le \min\left\{v_{j-1}^m, v_{j+1}^m\right\} \le v_j^{m+1} \le \max\left\{v_{j-1}^m, v_{j+1}^m\right\} \le \max_j v_j^m.$$

Assume from now on that r satisfies this condition.

4b

Assume that w_j^m solves

$$L_{\Delta x} w_j^m = g_j^m$$

for $m \ge 0$ and $j = 0, \ldots, N$ with periodic boundary conditions $w_{-1}^m = w_N^m$, $w_{N+1}^m = w_0^m$. Here g_j^m is a given grid function. We assume that $w_j^0 = 0$ for all j. Show that

$$\max_{j=0,\ldots,N} \left|w_j^m\right| \le m\Delta t \max_{\substack{j=0,\ldots,N\\ k=0,\ldots,m-1}} \left|g_j^k\right|.$$

Løsningsforslag: For $m = 0$ the estimate holds. Assume that it holds for m, then

$$\begin{aligned}
\left|w_j^{m+1}\right| &\le \left|\frac{1}{2}(1-r)\, w_{j+1}^m + \frac{1}{2}(1+r)\, w_{j-1}^m\right| + \Delta t \left|g_j^m\right| \\
&\le \max_{j=0,\ldots,N} \left|w_j^m\right| + \Delta t \left|g_j^m\right| \\
&\le m\Delta t \max_{\substack{j=0,\ldots,N\\ k=0,\ldots m-1}} \left|g_j^k\right| + \max_{j=0,\ldots,N} \left|g_j^m\right| \\
&\le (m+1)\Delta t \max_{\substack{j=0,\ldots,N\\ k=0,\ldots,m}} \left|g_j^k\right|.
\end{aligned}$$

4c

Let u be a smooth solution of (1), show that

$$L_{\Delta x} u(x_j, t^m) = \mathcal{O}(\Delta x),$$

and use this to obtain a bound of the error

$$\max_{j=0,\ldots,N} \left|v_j^m - u(x_j, t^m)\right|.$$

(Continued on page 7.)

Løsningsforslag: We have that (with $u_j^m = u(x_j, t^m)$)

$$
\begin{aligned}
u_j^m &= \frac{1}{2}\left(u_{j+1}^m + u_{j-1}^m\right) + \mathcal{O}(\Delta x^2), \\
u_t(x_j, t^m) &= \frac{1}{\Delta t}\left(u_j^{m+1} - u_j^m\right) + \mathcal{O}(\Delta t) \\
&= \frac{1}{\Delta t}\left(u_j^{m+1} - \frac{1}{2}\left(u_{j+1}^m + u_{j-1}^m\right) + \mathcal{O}(\Delta x^2)\right) + \mathcal{O}(\Delta t), \\
u_x(x_j, t^m) &= \frac{1}{2\Delta x}\left(u_{j+1}^m - u_{j-1}^m\right) + \mathcal{O}(\Delta x^2).
\end{aligned}
$$

Since $\Delta t = r\Delta x$, $\mathcal{O}(\Delta x) = \mathcal{O}(\Delta t)$, using the above we find that

$$L_{\Delta x} u_j^m = u_t + u_x + \mathcal{O}(\Delta x).$$

Define $e_j^m = u_j^m - v_j^m$, we find that

$$
\begin{cases}
L_{\Delta x} e_j^m = \mathcal{O}(\Delta x), \\
e_j^0 = 0.
\end{cases}
$$

By **b)**, we get

$$\max_j \left|e_j^m\right| \le t^m \mathcal{O}(\Delta x).$$

THE END