Math 3360 Spring 2024 Oblig 1

Submission deadline

Thursday 29 February 2024, 14:30 through Canvas.

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). **The assignment must be submitted as a single PDF file.** Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all the necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. (Add the code to the single pdf.) You can use your programming language of choice.

There is only one attempt to pass the assignment and you must have a score of at least 60% to pass it.

Application for postponement

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail:`studieinfo@math.uio.no`) well before the deadline.

Both mandatory assignments in this course must be approved in the same semester before you are allowed to take the final examination.

Complete guidelines on compulsory assignments

For further details on the hand in of compulsory assignments, see:
`https://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html`

Exercises

Exercise 1.

Solve the following PDE using the method of characteristics:

a)

$$\begin{aligned} u_t(x,t) + x u_x(x,t) = t^2 \qquad & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) = \exp(x) \qquad & x \in \mathbb{R}. \end{aligned}$$

b)

$$\begin{aligned} u_t(x,t) + x^2 u_x(x,t) = 0 \qquad & x > 0, \quad t > 0 \\ u(x,0) = \sin(x) \qquad & x > 0. \end{aligned}$$

(Note that the problem domain for x is different in b) than in a).)

Background for exercise 2:

For a set/sequence of numerical approximations $\{y_h\}_{h \geq 0}$ with $y_h \to \bar{y}$ as $h \to 0$, the big-O notation $y_h - \bar{y} = \mathcal{O}(h^\alpha)$, or equivalently $y_h = \bar{y} + \mathcal{O}(h^\alpha)$ denotes that there exist a constant $c > 0$ such that

$$|y_h - \bar{y}| \leq c h^\alpha \qquad \text{for all sufficiently small } h > 0.$$

The rate of convergence for $y_h \to \bar{y}$ is defined as the largest constant $\alpha > 0$ such that $y_h - \bar{y} = \mathcal{O}(h^\alpha)$.

For example, the sequence $y_h = h$ converges to $\bar{y} = 0$ with rate $\alpha = 1$ and the sequence $y_h = e^{-h^2}$ converges to $\bar{y} = 1$ with rate $\alpha = 2$, as

$$|y_h - 1| = |\int_0^{h^2} e^{-s} ds| \leq 1 \int_0^{h^2} ds = h^2.$$

Exercise 2.

When approximating derivatives of a function with the finite difference method, one cannot always use high-order central difference approximations since one does not always have access to function values both to the right and to the left of the approximation point. This exercise shows how one may proceed to obtain a high-order finite difference method in such settings.

a) Find values of the constants $a, b, c, d \in \mathbb{R}$ such that $u \in C^4(\mathbb{R})$ satisfies that

$$e_h := \left| \frac{a u(x) + b u(x+h) + c u(x+2h) + d u(x+3h)}{h^2} - u''(x) \right| = \mathcal{O}(h^2)$$

for any fixed $x \in \mathbb{R}$ and $\alpha = 2$. (The property "fixed $x \in \mathbb{R}$" enters in the bound through the constant in $\mathcal{O}$ may depending on x.)

Hint: Apply the mean-value theorem and Taylor expansion of the functions around the point x up to order 4 to obtain a linear system of equations for a, b, c and d.

b) Verify numerically that the difference approximation

$$\frac{au(x) + bu(x+h) + cu(x+2h) + du(x+3h)}{h^2} \approx u''(x)$$

you obtain in a) satisfies $e_h = \mathcal{O}(h^2)$ at $x = 1$ when applied to the function

$$u(x) = \tan(x) .$$

You can for instance proceed as follows: Suppose that

$$e_h = \left| u''(x) - \frac{au(x) + bu(x+h) + cu(x+2h) + du(x+3h)}{h^2} \right| \approx \tilde{c}h^2 \tag{1}$$

for some $\tilde{c} > 0$ for sufficiently small $h > 0$. Then

$$2 \approx \frac{\log(e_{h/2}/e_h)}{\log(1/2)}. \tag{2}$$

So for a sequence of values $h = 2^{-4}, 2^{-5}, \dots, 2^{-10}$, use subsequent h-values and formula (2) to verify that $e_h \to 0$ as $h \to 0$ has rate of convergence 2. The rate may also be estimated from slope of the (h, e_h) curve in a loglog plot, see this link for more.

Exercise 3.

At each point $x \in (0, 1)$ an elastic string subjected to a force

$$f(x; \lambda) = -\lambda \exp(-\lambda x) ,$$

where $\lambda > 0$ is a force parameter that can be adjusted. The string is fixed at the endpoints and the displacement from its stationary state 0 at the point $x \in (0, 1)$ is given by $u(x)$.

For a given $\lambda > 0$, the string displacement is described by the differential equation

$$\begin{aligned} -u''(x) &= f(x; \lambda) \qquad x \in (0, 1) \\ u(0) &= u(1) = 0, \end{aligned} \tag{3}$$

which has a unique solution in $C_0^2[0, 1] := \{g \in C^2[0, 1] \mid g(0) = g(1) = 0\}$.

a) Show that $u(x) \leq 0$ for all $x \in (0, 1)$.

Hint: Use for instance the Green's function solution representation and properties of the Green's function (or find the explicit solution).

b) To solve equation (3) numerically we introduce $h = 1/(n + 1)$, the uniform mesh $x_i = ih$ for $i = 0, 1, \dots, n + 1$, the difference approximation

$$-\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} = f(x_j; \lambda), \qquad j = 1, 2, \dots, n \tag{4}$$

and the boundary conditions $v_0 = v_{n+1} = 0$.

Write (4) as a system of linear equations $Av = b$ and solve the differential equation numerically using $n = 100$ for the parameter value $\lambda = 1$. Plot the numerical solution and compute the maximal displacement:

$$\|v\|_{h,\infty} := \max_{j=0,1,\dots,n+1} |v_j|.$$

c) Find a $\lambda \in [1, 10]$ such that the force $f(\cdot; \lambda)$ produces a maximal displacement that equals $1/10$ in the numerical solution of the ODE (or in the exact solution, if you like). You may for instance solve this through numerically approximating the value of $\lambda > 0$ such that the maximal displacement of the numerical solution of the differential equation (3) equals $1/10$.

Comment: For a numerical solution, any approach leading to a good approximation of λ is fine. You may for instance use the quite coarse resolution $n = 100$.

Exercise 4.

The following example shows that uniqueness of solutions can be a subtle property for nonlinear differential equations.

We consider the nonlinear ODE

$$\begin{aligned} -\frac{d}{dx}\big(u(x)u'(x)\big) &= 1 \qquad x \in (0,1) \\ u(0) = 0, &\quad u(1) = 0, \end{aligned} \tag{5}$$

where we seek solutions in $C_0^2(0,1) := \{g \in C^2(0,1) \cap C[0,1] \mid g(0) = g(1) = 0\}$.

a) Explain why the ODE is nonlinear. For simplicity, verify that the associated differential operator is nonlinear as a mapping with the following domain: $\mathcal{L} : C^2(0,1) \to C(0,1)$.

b) Show for instance by integration that the ODE has at least two solutions in $C_0^2(0,1)$. Can you also explain why it has exactly two solutions $C_0^2(0,1)$?

   Hint: $uu' = (u^2)'/2$.

c) Show that for any constant $a \neq 0$, the slightly changed ODE

$$\begin{aligned} -\frac{d}{dx}\big((a + u(x))u'(x)\big) &= 1 \qquad x \in (0,1) \\ u(0) = 0, &\quad u(1) = 0, \end{aligned} \tag{6}$$

   has a unique solution in $C_0^2[0,1]$ (and also $C_0^2(0,1)$, if you like).