Math 3360 Spring 2024 Oblig 2

Submission deadline

Thursday 2 May 2024, 14:30 through Canvas.

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). **The assignment must be submitted as a single PDF file.** Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all the necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. (Add the code to the single pdf.) You can use your programming language of choice.

There is only one attempt to pass the assignment and you must have a score of at least 60% to pass it.

Application for postponement

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail:`studieinfo@math.uio.no`) well before the deadline.

Both mandatory assignments in this course must be approved in the same semester before you are allowed to take the final examination.

Complete guidelines on compulsory assignments

For further details on the hand in of compulsory assignments, see:
`https://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html`

Exercises

Exercise 1.

This exercise explores a connection between the heat equation with a source term and the Poisson ODE, and how this can be used to numerically solve some nonlinear ODEs.

a) For

$$\begin{aligned} u_t &= u_{xx} + 1 \qquad & x \in (0,1), \quad t > 0 \\ u(0,t) = u(1,t) &= 0 & t \geq 0 \\ u(x,0) &= f(x) & x \in (0,1) \end{aligned} \tag{1}$$

with $f \in C_0^2[0,1]$, show that the PDE has at most one solution in $C^{2,1} := C^{2,1}([0,1] \times [0,\infty))$.

b) Assume there exists a unique solution $u \in C^{2,1}$ to (1), and let v be the unique solution to the Poisson equation

$$\begin{aligned} -v''(x) &= 1 \qquad x \in (0,1) \\ v(0) = v(1) &= 0, \end{aligned} \tag{2}$$

namely $v(x) = x(1-x)/2$. Prove the following mean square convergence result:

$$\lim_{t\to\infty} \int_0^1 (u(x,t) - v(x))^2 dx = 0.$$

We say that v is the stationary solution of the heat equation.

Hint: Consider the energy

$$E(t) = \int_0^1 (u(t,x) - v(x))^2 dx$$

and utilize Poincare's inequality (see Lemma 8.6 in TW):

$$\int_0^1 (u(x,t) - v(x))^2 dx \leq \frac{1}{2} \int_0^1 (u_x(x,t) - v'(x))^2 dx$$

to obtain a differential inequality for $E(t)$.

c) Test numerically if $u(\cdot,t)$ converges to v in mean square sense as $t \to \infty$ for the setting described in part b) by solving (1) with $f \equiv 0$ with the explicit scheme up to final times $T = 1/8, 1/4, 1/2, 1$ (admittedly this is quite far from t going to infinity, but it should be large enough to see a trend for this problem). Plot the solutions together with $v(x)$ to illustrate the convergence.

d) We recall from Oblig 1 exercise 4c) that the nonlinear ODE

$$\begin{aligned} -((0.1 + v)v')'(x) &= 1 \\ v(0) = v(1) &= 0, \end{aligned} \tag{3}$$

has the unique solution $v(x) = \sqrt{x(1-x) + 0.01} - 0.1$. The nonlinearity in the ODE makes it more complicated to solve the this problem numerically than solving linear

ODE such as the Poisson equation. In this exercise we will extend the explicit-scheme approach in part c) to this setting: Construct an explicit numerical scheme solving the PDE

$$\begin{aligned} u_t &= ((0.1+u)u_x)_x + 1 \qquad && x \in (0,1), \quad t > 0 \\ u(0,t) = u(1,t) &= 0 && t \geq 0 \\ u(x,0) &= 0 && x \in (0,1). \end{aligned} \tag{4}$$

and investigate if your numerical solution converges to v when $t \to \infty$ (for instance by plotting your numerical solution for a sequence of t-values and $v(x)$).

Hint: See e.g. TW exercise 4.20 for inspiration to construct a scheme, and impose the stability constraint $\Delta t/\Delta x^2 \leq 1/4$.

Exercise 2.

Consider the problem setting of Tveito and Winter Exercise 4.15 part a). For a given set of mesh points (x_j, t_m), let $u_j^m := u(x_j, t_m)$ where $u(x,t)$ denotes the exact solution of the continuous problem

$$\begin{aligned} u_t &= u_{xx} \qquad && x \in (0,1), \quad t \in (0,T] \\ u(0,t) = u(1,t) &= 0 && t \in [0,T] \\ u(x,0) &= f(x) && x \in (0,1) \end{aligned} \tag{5}$$

For the truncation error

$$\tau_j^m = \frac{u_j^{m+1} - u_j^m}{\Delta t} - \frac{u_{j-1}^m - 2u_j^m + u_{j+1}^m}{\Delta x^2}$$

we assume there exists a constant $C > 0$ such that

$$|\tau_j^m| \leq C(\Delta t + \Delta x^2)$$

for all $(x_j, t_m) \in (0,1) \times [0,T)$, where the mesh takes the standard form $x_j = j\Delta x$ with $\Delta x = 1/(n+1)$ and $t_m = m\Delta t$ for some $\Delta t > 0$.

a) Let $v_j^m = v(x_j, t_m)$ denote the explicit-scheme numerical solution of the PDE (5), meaning the discrete function that satisfies

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} - \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{\Delta x^2} = 0 \tag{6}$$

for all grid points $(x_j, t_m) \in (0,1) \times [0,T)$, $v(0,t_m) = v(1,t_m) = 0$ for all grid points $t_m \in [0,T]$ and $v(x_j, 0) = f(x_j)$ for all grid points $x_j \in (0,1)$.

Show that if $r := \Delta t/\Delta x^2 \leq 1/2$, then it holds for all grid points $t_m \in [0,T]$ that

$$e_m := \max_{j=0,1,\ldots,n+1} |u_j^m - v_j^m| \leq \frac{3}{2} C t_m \Delta x^2$$

Hint: Exploit that

$$u_j^{m+1} = u_j^m + r\big(u_{j-1}^m - 2u_j^m + u_{j+1}^m\big) + \Delta t \tau_j^m$$

and try to combine with (6).

b) Test the result in part a) numerically at the time $t = 1/10$ on the PDE (5) with $f(x) = 10\sin(2\pi x)$, which has the unique solution $u(x,t) = 10\exp(-(2\pi)^2 t)\sin(2\pi x)$. Try to verify the rate 2 in Δx in your numerical experiments using for example a loglog plot.

Exercise 3.

We consider the annulus $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4\}$ with boundary $\partial\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$, and the Laplace equation

$$\begin{aligned} u_{xx} + u_{yy} &= 0 & (x,y) &\in \Omega \\ u(x,y) &= f(x,y) & (x,y) &\in \partial\Omega. \end{aligned}$$

a) Show that for any solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$, it holds that

$$\sup_{(x,y)\in\Omega} |u(x,y)| \leq \sup_{(x,y)\in\partial\Omega} |f(x,y)|$$

b) Find the unique solution to the above PDE when, in polar coordinates,

$$f(r,\theta) = \begin{cases} \sin(\theta) + 7\cos(2\theta) & \text{if} \quad r = 1 \\ 0 & \text{if} \quad r = 2. \end{cases}$$

Hint: Rewrite the PDE in polar coordinates and determine the solution going through the steps for finding a formal solution.