For $\beta > 0$, the general solution for the ODE

$$u''(x) = -\beta^2 u(x) \qquad x \in [0,1] \quad (1)$$

is $u(x) = c_1 \sin(\beta x) + c_2 \cos(\beta x)$

<u>Proof</u>

Let $v_1(x) := u(x)$ and

$v_2(x) := u'(x)$

and write $v(x) = \begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix}$.

Then $v_1' = u' = v_2$

& $v_2' = u'' = -\beta^2 u = -\beta^2 v_1$

$\Rightarrow$

$$v' = \underbrace{\begin{bmatrix} 0 & 1 \\ -\beta^2 & 0 \end{bmatrix}}_{=:A} v \qquad (*)$$

(where $v' = \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}$).

Diagonalization of A:

$$A = \begin{bmatrix} 0 & 1 \\ -\beta^2 & 0 \end{bmatrix}$$ has eigvals

$$\lambda_1 = i\beta \quad \lambda_2 = -i\beta$$

with eigenvectors

$$w_1 = \begin{bmatrix} -i \\ \beta \end{bmatrix} \quad w_2 = \begin{bmatrix} i \\ \beta \end{bmatrix}$$

(**)

The matrix

$$W = \begin{bmatrix} -i & i \\ \beta & \beta \end{bmatrix}$$ is invertible

$$\left(W^{-1} = \frac{1}{2i\beta} \begin{bmatrix} -\beta & i \\ \beta & i \end{bmatrix} \right)$$ and

$$AW = [Aw_1 \;\; Aw_2] \overset{(**)}{=} [\lambda_1 w_1 \;\; \lambda_2 w_2]$$

$$= W \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{=:\Lambda}$$

Multiply from right with W^{-1} in

$$AW = W\Lambda \quad \Big| \cdot W^{-1}$$

$$\Rightarrow \qquad A = W\Lambda W^{-1}$$

(so A is diagonalizable).

Returning to ODE $(*)$

$$v' = Av$$

$$\Rightarrow \quad W^{-1}v' = \Lambda W^{-1} v \qquad (***)$$

Set

$$\tilde{v}(x) := W^{-1} v(x) = W^{-1} \begin{bmatrix} v_1(x) \\ v_2(x) \end{bmatrix}.$$

Then, since $\frac{d}{dx}\tilde{v} = W^{-1}v'$

$$\tilde{v}' = \Lambda \tilde{v} = \begin{bmatrix} \lambda_1 \tilde{v}_1 \\ \lambda_2 \tilde{v}_2 \end{bmatrix}$$

with general complex-valued solution

$$\tilde{V}_1(x) = e^{i\beta x}\tilde{c}_1$$
$$\tilde{V}_2(x) = e^{-i\beta x}\tilde{c}_2 \quad , \quad \tilde{c}_1, \tilde{c}_2 \in \mathbb{C},$$

(since you can solve ODE components individually).

And

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = W \begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} -i\tilde{c}_1 e^{i\beta x} + i\tilde{c}_2 e^{-i\beta x} \\ \beta\tilde{c}_1 e^{i\beta x} + \beta\tilde{c}_2 e^{-i\beta x} \end{bmatrix}$$

Recalling that

$V_1(x) = u(x)$ the general complex-valued solution to (1) is

$$u(x) = (-i\tilde{c}_1 + i\tilde{c}_2)\cos(\beta x) + (\tilde{c}_1 + \tilde{c}_2)\sin(\beta x)$$

where $\tilde{c}_1, \tilde{c}_2 \in \mathbb{C}$.

Set $c_1 = \text{real}(i(\tilde{c}_2 - \tilde{c}_1))$
$\qquad = \text{imag}(\tilde{c}_1 - \tilde{c}_2)$

& $c_2 = \text{real}(\tilde{c}_1 + \tilde{c}_2)$

(choosing e.g. $\tilde{C}_2 = 0$ and $\tilde{C}_1$ any value in $\mathbb{C}$, it is clear that the associated (C_1, C_2) can take any value in $\mathbb{R}^2$.)

So general real-valued solution to the ODE (1) becomes

$$u(x) = C_1 \cos(\beta x) + C_2 \sin(\beta x)$$

$$C_1, C_2 \in \mathbb{R}.$$

$\boxed{\Xi}$